Left and right convergence of graphs with bounded degree

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Contents

Abstract

The theory of convergent graph sequences has been worked out in two extreme cases, dense graphs and bounded degree graphs. One can define convergence in terms of counting homomorphisms from fixed graphs into members of the sequence (left-convergence), or counting homomorphisms into fixed graphs (right-convergence). Under appropriate conditions, these two ways of defining convergence was proved to be equivalent in the dense case by Borgs, Chayes, Lovász, Sós and Vesztergombi. In this paper a similar equivalence is established in the bounded degree case.

In terms of statistical physics, the implication that left convergence implies right convergence means that for a left-convergent sequence, partition functions of a large class of

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statistical physics models converge. The proof relies on techniques from statistical physics, like cluster expansion and Dobrushin Uniqueness.

1   Introduction

The theory of convergent graph sequences has been worked out in two extreme cases, dense graphs and bounded degree graphs. The case of dense graphs is probably easier; convergence of such graphs was introduced and characterized in different ways in [5, 6]. Convergence of bounded degree graph sequences was defined by Benjamini and Schramm [2], and has inspired a lot of work [3, 15, 8, 22]. While this is perhaps the more important case from the point of view of applications, the theory is less complete, and, for example, some of the characterizations of convergence in the dense case have no analogues. The goal of this paper is to prove, for bounded degree graphs, an analogue of one of these characterizations.

For a simple finite graph G and node $v \in V(G)$, let $B_G(v, r)$ denote the subgraph of G induced by the nodes at distance at most r from v. We consider this as a rooted graph, with v designated as its root. Following [2], a sequence $(G_1, \ldots, G_n)$ of simple graphs with degrees uniformly bounded by D is called *locally convergent* (weakly convergent, left-convergent) if for every fixed integer $r \geq 1$, selecting a random node v uniformly from $V(G)$, the probability that $B_{G_n}(v, r)$ is a fixed rooted graph U tends to a limit as $n \to \infty$. It is not hard to see that this definition is equivalent to saying that for every connected graph F, $\mathrm{hom}(F, G_n)/|G_n|$ tends to a limit, where $\mathrm{hom}(F, G_n)$ is the number of homomorphisms (adjacency preserving maps) of F into G_n, and $|G_n| = |V(G_n)|$ is the number of nodes of G_n.

This reformulation raises the possibility of defining convergence by turning the arrows around, i.e., in terms of $\mathrm{hom}(G_n, H)$ for fixed graphs H. It is easy to see that the "right" normalization in this case is $\ln \mathrm{hom}(G_n, H)/|G_n|$.

In general, convergence of $\ln \mathrm{hom}(G_n, H)/|G_n|$ does not follow from left-convergence. As an example, if G_n is the n-cycle, then the sequence (G_n) is trivially locally convergent. But if K_2 is the complete graph with 2 nodes, then $\ln \mathrm{hom}(G_n, K_2)/|G_n|$ alternates between $(\ln 2)/n$ and $-\infty$ depending on the parity of n.

We will prove (Theorem 3.1 that if H is sufficiently dense (depending on D), then local convergence does imply that $\ln \mathrm{hom}(G_n, H)/|G_n|$ is convergent. Conversely (4.3), if $\ln \mathrm{hom}(G_n, H)/|G_n|$ is convergent for every sufficiently dense H, then (G_n) is locally convergent.

2   Preliminaries

A graph is *simple* if it has no loops or parallel edges. A *simple looped graph* is a simple graph with a loop added at each of its node. A *weighted graph H* is a graph with a weight $\alpha_i(H) > 0$ associated with each node i and a real weight $\beta_{ij}(H)$ associated with each edge ij. (The weights $\beta_{ij}(H)$ may be negative.) We will consider each weighted graph as a complete graph with loops at all nodes; absence of an edge is indicated by weight 0. (No parallel edges are allowed in a weighted graph.) If the graph H is understood from the context, we use the notation α_i and

β_{ij}. We can consider every simple graph (looped or not looped) as a weighted graph on the same node set, where the original edges have weight 1 and the missing edges have weight 0. Let $\alpha_H = \sum_{i \in V(H)} \alpha_i(H)$ denote the total nodeweight of H.

We denote by $\overline{H}$ the weighted graph obtained from H by replacing each edgeweight β_{ij} by $1 - \beta_{ij}$. (Note that this also applies to loops. So if H is the weighted graph corresponding to a simple graph, then $\overline{H}$ corresponds to the complement of H with a loop added at each node.)

Let $\mathrm{Sub}(G)$ denote the set of nonempty subgraphs of G without isolated nodes, $\mathrm{Con}(G)$, the set of connected subgraphs of G with at least two nodes, $\mathrm{CInd}(G)$, the set of connected induced subgraphs of G with at least two nodes, and $\mathrm{CSpan}(G)$, the set of connected spanning subgraphs of G (with one or more nodes).

To simplify notation, we will write $|G| = |V(G)|$.

Let $\{A_1, \ldots, A_n\}$ be a family of sets. We denote by $L(A_1, \ldots, A_n)$ their intersection graph, i.e., the graph on $V = [n]$ in which we connect i and j by an edge if $A_i \cap A_j \neq \emptyset$.

For a set V, we denote by $\Pi(V)$ the set of its partitions.

2.1 Homomorphism numbers and densities

For two (finite) simple graphs F and G, $\mathrm{hom}(F, G)$ denotes the number of homomorphisms (adjacency preserving maps) from F to G. We define $t(F, G)$ to be the probability that a random map of $V(F)$ into $V(G)$ is a homomorphism, i.e.,

$$t(F, G) = \frac{\mathrm{hom}(F, G)}{|G|^{|F|}}.$$

We call $t(F, G)$ the *homomorphism density* of F in G.

Sometimes we need to consider the number of injective homomorphisms $\mathrm{inj}(F, G)$, the number of embeddings as induced subgraphs, $\mathrm{ind}(F, G)$, and the number of homomorphisms surjective on both the nodes and the edges, $\mathrm{surj}(F, G)$. We denote by $\mathrm{aut}(F) = \mathrm{ind}(F, F) = \mathrm{surj}(F, F)$ the number of automorphisms of F. The quotients $\mathrm{inj}_0(F, G) = \mathrm{inj}(F, G)/\mathrm{inj}(F, F)$ and $\mathrm{ind}_0(F, G) = \mathrm{ind}(F, G)/\mathrm{ind}(F, F)$ count the numbers of subgraphs and induced subgraphs of G isomorphic to F, respectively.

We extend these notions to the case when the target graph (denoted by H in this case) is weighted:

$$\mathrm{hom}(F, H) = \sum_{\phi: \ V(F) \to V(H)} \prod_{u \in V(F)} \alpha_{\phi(u)}(H) \prod_{uv \in E(F)} \beta_{\phi(u), \phi(v)}(H)$$

where the sum runs over all maps from $V(F)$ to $V(HG)$. The homomorphism density is now defined as

$$t(F, H) = \frac{\mathrm{hom}(F, H)}{\alpha_H^{|F|}}.$$

We will also need the following related quantity:

$$z(G, H) = \sum_{F \in \mathrm{CSpan}(G)} (-1)^{|E(F)|} t(F, H). \tag{1}$$

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This quantity plays an important role in the sequel, and it would be interesting to explore its combinatorial significance. We have the inverse relation

$$\sum_{F \in \mathrm{CSpan}(G)} (-1)^{|E(F)|} z(F, H) = \sum_{F \in \mathrm{CSpan}(G)} (-1)^{|E(F)|} \sum_{J \in \mathrm{CSpan}(F)} (-1)^{|E(J)|} t(J, H)$$

$$= \sum_{J \in \mathrm{CSpan}(G)} t(J, H) \sum_{J \subseteq F \subseteq G} (-1)^{|E(F)| - |E(J)|} = t(G, H). \tag{2}$$

2.2 Local convergence of a graph sequence

Let $G = (V, E)$ be a graph with degrees bounded by D, and fix an integer $r \geq 0$. For $v \in V$, let $N(v) = N_G(v)$ denote set of nodes adjacent to v. For $v \in V$, let $B(v, r) = B_G(v, r)$ denote the subgraph of G induced by all nodes at a distance at most r from v (the *r-neighborhood* of node v, or the *r-ball* about v). So $B(v, 1)$ is the subgraph induced by $N(v) \cup \{v\}$. We consider $B(v, r)$ as a rooted graph, i.e., the node v is specified as the center of $B(v, r)$. For fixed r, there is a finite number (depending on D and r only) of possible r-balls $U_1, \ldots, U_N$. Let $\mu(G, U_i)$ denote the fraction of nodes of G whose r-neighborhood is U_i. We can think of μ as a probability distribution on possible r-balls.

We call a sequence (G_n) of graphs with degrees bounded by D *locally convergent*, or *left-convergent* if $\mu(G_n, U)$ tends to a limit $\mu(U)$ as $n \to \infty$ for every r and every r-ball U.

It is easy to see that if (G_n) is left-convergent, then for every connected graph F the sequence $\hom(F, G_n)/|G_n|$ is convergent. It is also quite easy to see that this property is sufficient for local convergence. By formulas (1) and (2), we could also require that $z(F, G_n)$ is convergent for every simple graph F.

We can also talk about *right-convergence*, meaning that $\ln \hom(G_n, H)/|G_n|$ tends to a limit as $n \to \infty$ (here H can be a weighted graph; we'll see that the normalization is appropriate). However, we will not formally define right-convergence, since there seem to be different ways of specifying which weighted graphs H to consider here. Rather, we want to find a reasonable class of weighted graphs H for which right-convergence is equivalent to local (left-) convergence.

2.3 Chromatic polynomial

Let $G = (V, E)$ be a simple graph with n nodes. For every nonnegative integer y, we denote by $\mathrm{chr}(G, y)$ the number of y-colorations of G. Note that $\mathrm{chr}(G, q) = \hom(G, K_q)$.

Let $\mathrm{chr}_0(G, k)$ denote the number of k-colorations of G in which all colors occur. Then clearly

$$\mathrm{chr}(G, y) = \sum_{k=0}^{\infty} \mathrm{chr}_0(G, k) \binom{y}{k}. \tag{3}$$

This implies that $\mathrm{chr}(G, y)$ is a polynomial in y with leading term y^n and constant term 0, which is called the *chromatic polynomial* of G. It is easy to see that if G is a simple graph, then for every $e \in E(G)$,

$$\mathrm{chr}(G, q) = \mathrm{chr}(G \setminus e, q) - \mathrm{chr}(G/e, q), \tag{4}$$

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where $G \setminus e$ and G/e arise from G by deleting and contracting e, respectively (in G/e, parallel edges are collapsed to one). From this recurrence a number of properties of the chromatic polynomial are easily proved, for example, that its coefficients alternate in sign.

The coefficient of the linear term in the chromatic polynomial is

$$\mathsf{cr}(G) = \sum_{G' \in \mathrm{CSpan}} (-1)^{|E(G')|}, \tag{5}$$

which is called the *Crapo invariant* or *chromatic invariant* of the graph. Trivially, $\mathsf{cr}(G) = 0$ if G is disconnected. It follows from (4) that if G is a simple graph, then for every $e \in E(G)$,

$$\mathsf{cr}(G) = \mathsf{cr}(G \setminus e) - \mathsf{cr}(G/e). \tag{6}$$

This implies by induction that $(-1)^{|G|-1}\mathsf{cr}(G) > 0$ if G is connected.

2.4   Subtree counts

Let $\mathsf{tree}(G)$ denote the number of spanning trees in G. More generally, let $\mathsf{tree}(G; v, k)$ denote the number of subtrees of G with k nodes, containing a given node $v \in V(G)$. Let T_D be an infinite rooted D-ary tree, with root r. The following formula is well known ([24], Theorem 5.3.10):

$$\mathsf{tree}(T_D; r, k) = \frac{1}{k}\binom{kD}{k-1}. \tag{7}$$

The right hand side has this more convenient estimate:

$$\frac{1}{k}\binom{kD}{k-1} \le \frac{(kD)^{k-1}}{k!} < \frac{e^k D^{k-1}}{k\sqrt{2\pi k}} < \frac{(eD)^{k-1}}{2}$$

(assuming $k \ge 3$, but the bound is trivially true for $k = 2$ as well).

Lemma 2.1 *Let G be a graph with maximum degree D and let $v \in V(G)$.*

(a) *The number of subtrees of G with k nodes containing v is at most $\frac{1}{k}\binom{kD}{k-1}$.*

(b) *The number of connected subgraphs of G with m edges containing v is at most $\frac{1}{m+1}\binom{(m+1)D}{m}$.*

(c) *The number of connected induced subgraphs of G with k nodes containing v is at most $\frac{1}{k}\binom{kD}{k-1}$.*

(d) *The number of connected subgraphs of G with k nodes containing v is at most 2^{Dk}.*

Proof.   Throughout this proof, we drop the subscript D from T_D.

(a) It suffices to note the easy fact that for any graph G with maximum degree D, the number of subtrees with k nodes containing v is not larger than the corresponding number in T. Indeed, we could replace T by the rooted tree in which the root has degree D, all the other nodes have (down-)degree $D - 1$.

(b) Similarly as in (a) it suffices to prove that for any graph G with maximum degree D, the number of connected subgraphs of G with m edges containing v is not larger than the corresponding number in T.

Let us label the edges going out of any given node v of G arbitrarily by $1, \ldots, d_G(v)$. Also, label the edges going from a node of T to its children arbitrarily $1, \ldots, D$. Let F be a connected subgraph of G containing v. Let T_0 be a spanning tree in F. Orient F so that T_0 is oriented away from the root v (the other edges are oriented arbitrarily).

There is a unique embedding $\phi : V(F) \hookrightarrow V(T)$ that preserves the edges of F, and for at each node $u \in V(F)$, the edges of T_0 leaving u in T_0 are mapped onto edges in T with the same label. For every edge $ab \in E(F) \setminus E(T_0)$, map it onto the edge of T leaving $\phi(a)$ with the same label. This assigns to F a subtree F' of T with m edges.

Clearly, F' uniquely determines F: starting from the root, we can map the edges of T back into G. This proves (b).

(c) is a trivial consequence of (a).

(d) We can select the edges of F incident with v in less than 2^D ways; then going to one of the neighbors v_1 of v, we can select the set of edges of F incident with v_1 in less than 2^D ways, etc. Repeating this k times, we have finished selecting F. $\qquad\square$

2.5 Weighted subtrees and weighted chromatic invariants

Let H be a weighted graph. We extend the definitions of the number of subtrees and of the chromatic invariant to weighted graphs:

$$\mathsf{cr}(H) = \sum_{F \in \mathrm{CSpan}(H)} (-1)^{|E(F)|} \prod_{e \in F} \beta_e, \qquad \mathsf{tree}(H) = \sum_{F \in \mathrm{SpTr}(G)} \prod_{e \in F} \beta_e.$$

We note that if all edgeweights of H are 1, then $\mathsf{tree}(H)$ is the number of spanning trees and $\mathsf{cr}(H)$ is the chromatic invariant of the underlying simple graph. (The nodeweights play no role in these definitions.)

For $e \in E(H)$, let $H - e$ denote the weighted graph obtained from H by deleting the edge e. We need two versions of the operation of contracting an edge. Let H/e denote the graph obtained by contracting e, where the arising parallel edges are replaced by a single edge whose weight is the sum of the weights of its pre-images. Let $H \div e$ denote the graph obtained from H similarly, except that the new edgeweight is the sum minus the product of the weights of its pre-images. (Note that for graphs H with edgeweights between 0 and 1, the resulting edgeweight again lies between 0 and 1, which is not necessarily the case for H/e). The two quantities introduced above satisfy the recurrence relations

$$\mathsf{cr}(H) = \mathsf{cr}(H - e) - \beta_e \mathsf{cr}(H \div e), \qquad \mathsf{tree}(H) = \mathsf{tree}(H - e) + \beta_e \mathsf{tree}(H/e). \tag{8}$$

For graphs with edge weights between 0 and 1, the first of these relations implies that $(-1)^{|H|-1}\mathsf{cr}(H) > 0$.

Let G be a simple graph and H, a weighted graph, and let $\tilde{\alpha}$ be the normalized weight $\tilde{\alpha}_i = \alpha_i(F)/\alpha_F$. By a *random map* $G \to H$ we mean a map $V(G) \to V(H)$, where the image of each node of G is chosen independently from the probability distribution $\tilde{\alpha}$.

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For any map $\phi : V(G) \to V(H)$, we can define a weighting of G, where the weight of an edge of G is the weight of its image in H. We denote this weighted graph by G^ϕ. Note that for a random map $G \to H$ we have

$$t(G, H) = \mathsf{E}_\phi \prod_{ij \in E(G)} \beta_{\phi(i)\phi(j)},$$

and so

$$
\begin{aligned}
z(G, H) &= \sum_{F \in \mathrm{CSpan}(G)} (-1)^{|E(F)|} t(F, H) \\
&= \sum_{F \in \mathrm{CSpan}(G)} (-1)^{|E(F)|} \mathsf{E}_\phi \prod_{ij \in E(F)} \beta_{\phi(i)\phi(j)} = \mathsf{E}_\phi \mathsf{cr}(G^\phi).
\end{aligned}
\tag{9}
$$

Lemma 2.2 *Let H be a weighted graph with node weights 1 and edge weights in $[0,1]$, then*

$$|\mathsf{cr}(H)| \leq \mathsf{tree}(H).$$

Proof. By equation (8),

$$|\mathsf{cr}(H)| \leq |\mathsf{cr}(H - e)| + \beta_e(H)|\mathsf{cr}(H \div e)|.$$

Since the edgeweights in $H \div e$ are not larger than the corresponding edgeweights in H/e, we get by induction on the number of edges that

$$|\mathsf{cr}(H)| \leq \mathsf{tree}(H - e) + \beta_e(H)\mathsf{tree}(H/e) = \mathsf{tree}(H).$$

$\square$

To state our next lemma, we define

$$c(H) = \max_{u \in V(H)} \sum_{v \in V(H)} \frac{\alpha_v}{\alpha_H} |\beta_{uv}|.$$

Lemma 2.3 *Let G be a simple graph, and let H be a weighted graph. Let ϕ be a random map $G \to H$. Then*

$$\mathsf{E}_\phi |\mathsf{tree}(G^\phi)| \leq \mathsf{tree}(G) c(H)^{|G|-1}.$$

Proof. We may assume the edgeweights in H are nonnegative. We have

$$\mathsf{E}_\phi(\mathsf{tree}(G^\phi)) = \mathsf{E}_\phi \Big(\sum_{T \in \mathrm{SpTr}(G)} \prod_{ij \in E(T)} \beta_{\phi(i)\phi(j)} \Big) = \sum_{T \in \mathrm{SpTr}(G)} \mathsf{E}_\phi \Big(\prod_{ij \in E(T)} \beta_{\phi(i)\phi(j)} \Big) \tag{10}$$

Fix the tree T, and let p be one of its endpoints, with single neighbor q. Then picking the random map ψ of $V(G) \setminus \{p\}$ first and the the image u of p last, we get

$$
\begin{aligned}
\mathsf{E}_\phi \Big(\prod_{ij \in E(T)} \beta_{\phi(i)\phi(j)} \Big) &= \mathsf{E}_\psi \Big(\prod_{ij \in E(T-p)} \beta_{\phi(i)\phi(j)} \mathsf{E}_u(\beta_{\psi(q)u}) \Big) \\
&= \mathsf{E}_\psi \Big(\prod_{ij \in E(T-p)} \beta_{\phi(i)\phi(j)} \sum_u \frac{\alpha_u}{\alpha_H} \beta_{\psi(q)u} \Big) \\
&\leq \mathsf{E}_\psi \Big(\prod_{ij \in E(T-p)} \beta_{\phi(i)\phi(j)} c(H) \Big) = c(H) \mathsf{E}_\psi \Big(\prod_{ij \in E(T-p)} \beta_{\phi(i)\phi(j)} \Big),
\end{aligned}
$$

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whence by induction

$$\mathsf{E}_\phi \Big(\prod_{ij \in E(T)} \beta_{\phi(i)\phi(j)} \Big) \le c(H)^{|T|-1}.$$

By (10), the Lemma follows. □

Lemma 2.4 *Let G be a simple graph, and let H be a weighted graph with edge weights in $[0,1]$. Then*

$$|z(G,H)| \le \mathsf{tree}(G)c(H)^{|G|-1}.$$

Proof. Let ϕ be a random map $G \to T$. Then by (9) and Lemmas 2.2 and 2.3,

$$|z(G,H)| \le \mathsf{E}_\phi |\mathsf{cr}(G^\phi)| \le \mathsf{E}_\phi \mathsf{tree}(G^\phi) \le \mathsf{tree}(G)c(H)^{|G|-1}.$$

□

3 Left-convergence implies right-convergence

Our first main theorem is the following.

Theorem 3.1 *Let (G_n) be a left-convergent sequence of graphs with maximum degree at most D. Let H be a weighted graph with $0 \le \beta_{ij} \le 1$ and $c(\overline{H}) < 1/(2D)$. Then $\ln t(G_n,H)/|G_n|$ is convergent as $n \to \infty$.*

Recall that the condition on $c(\overline{H})$ means that

$$\sum_{k \in V(H)} \frac{\alpha_k}{\alpha_H}(1 - \beta_{ik}) < \frac{1}{2D} \tag{11}$$

for all $i \in V(H)$. It is clear that we may assume that $\alpha_H = 1$.

We give two proofs of this Theorem: one, using the Dobrushin Uniqueness Theorem, and another one using cluster expansion techniques. In fact, the second proves a weaker result only, where in (11), the $2D$ in the denominator is replaced by the stronger condition $8D$. The reason for giving it at all is that (a) it uses a completely different technique, (b) it gives approximation formulas for $\ln t(G,H)/|G|$ with explicit error bounds, and (c) the method can also be used to prove the converse of the theorem (see Theorem 4.3).

3.1 Proof via Dobrushin Uniqueness

We briefly recall two basic notions (see e.g. [12] for a more informative discussion):

(i) A *coupling* of probability distributions μ and ν is a random pair (X,Y) defined on some probability space such that the marginal distribution of X is μ and that of Y is ν. A coupling of (not necessarily real-valued) random variables ϕ, ψ is a random pair (X,Y) such that the laws of X and Y are those of ϕ and ψ (respectively).

(ii) The *total variation distance* between discrete probability distributions μ and ν on Ω is $\|\nu - \mu\|_{\mathrm{TV}} = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)|$; it is equal to the minimum over couplings (X, Y) of μ and ν of $\Pr(X \neq Y)$. The total variation distance of a pair of random variables ϕ, ψ is the total variation distance of their distributions.

For the rest of this section we fix H as in Theorem 3.1, and set $t(G, H) = \hom(G, H) = t(G)$. For any G and $\phi : V(G) \to V(H)$, set

$$W(\phi) = \prod_{u \in V(G)} \alpha_{\phi(u)} \prod_{uv \in E(G)} \beta_{\phi(u), \phi(v)}.$$

The natural associated probability measure on $V(H)^{V(G)}$ is given by $\Pr_G(\phi) \propto W(\phi)$ (that is, $\Pr_G(\phi) = W(\phi)/t(G)$). We write E_G for expectation with respect to this measure.

Given $\Lambda \subseteq V(G)$ and $\alpha : V(G) \setminus \Lambda \to V(H)$, let $\phi_\alpha : V(G) \to V(H)$ be chosen according to

$$\Pr(\phi_\alpha = \tau) = \Pr_G(\phi = \tau | \phi \equiv \alpha \text{ off } \Lambda) \qquad \forall \tau : V(G) \to V(H).$$

For $\zeta = (\zeta_1, \ldots, \zeta_s)$ with $\zeta_i \in V(H)$, define the probability distribution ν_ζ on $V(H)$ by

$$\nu_\zeta(i) \propto \alpha_i \prod_{j=1}^{s} \beta_{i, \zeta_j}.$$

(Thus ν_ζ is the conditional distribution of $\phi(v)$ given that $d(v) = s$ and the ϕ-values of the neighbors of v are $\zeta_1, \ldots, \zeta_s$.) The following version of Dobrushin Uniqueness is convenient for our purposes, but see e.g. [10] for a more usual statement.

Theorem 3.2 *Let G be a graph with maximum degree at most D, and let H be a weighted graph such that with notation as above, there is a $0 < \kappa < 1$ such that for any $s \leq D$ and $\zeta = (\zeta_1, \ldots, \zeta_s)$ and $\zeta' = (\zeta_1, \ldots, \zeta_{s-1}, \zeta_s')$ with $\zeta_1, \ldots, \zeta_s, \zeta_s' \in V(H)$, we have*

$$\|\nu_\zeta - \nu_{\zeta'}\|_{\mathrm{TV}} \leq \frac{\kappa}{D}. \tag{12}$$

Let $\Lambda \subseteq V(G)$, $\Lambda' = V(G) \setminus \Lambda$ and $\alpha, \beta : \Lambda' \to V(H)$. Then there is a coupling $(\tilde{\phi}_\alpha, \tilde{\phi}_\beta)$ of ϕ_α and ϕ_β such that

$$\Pr(\tilde{\phi}_\alpha \neq \tilde{\phi}_\beta) \leq \kappa^{d(x, \Lambda')} \qquad \forall x \in V(G).$$

In particular, for any $\Omega \subseteq \Lambda$ the total variation distance of the restrictions of ϕ_α and ϕ_β to Ω is at most $\sum_{x \in \Omega} \kappa^{d(x, \Lambda')}$.

To apply this theorem, we need a couple of simple facts.

Proposition 3.3 *Let $\gamma_i \geq \mu_i, \nu_i \geq 0$ for $i = 1, \ldots, n$, $\gamma = \sum \gamma_i$, $\mu = \sum \mu_i$, $\nu = \sum \nu_i$ and $\xi \geq 0$ and suppose $(\gamma \geq) \nu, \mu \geq \gamma - \xi$. Then*

$$\sum_{i=1}^{n} \left| \frac{\mu_i}{\mu} - \frac{\nu_i}{\nu} \right| \leq \frac{2\xi}{\gamma - \xi} \; ;$$

that is, the total variation distance of the distributions $\{\mu_i/\mu\}_{i \in [n]}$ and $\{\nu_i/\nu\}_{i \in [n]}$ is at most $\xi/(\gamma - \xi)$.

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Proof. Assuming (w.l.o.g.) that $\nu \geq \mu$, we have (with sums over $i \in [n]$),

$$\sum |\frac{\mu_i}{\mu} - \frac{\nu_i}{\nu}| \leq \sum \mu_i \left(\frac{1}{\mu} - \frac{1}{\nu}\right) + \frac{1}{\nu} \sum |\mu_i - \nu_i|$$
$$\leq \frac{\nu - \mu}{\nu} + \frac{1}{\nu} \sum ((\gamma_i - \mu_i) + (\gamma_i - \nu_i))$$
$$= 2\frac{\gamma - \mu}{\nu} \leq 2\frac{\xi}{\gamma - \xi}.$$

$\square$

Proposition 3.4 *The conditions on H in Theorem 3.1 imply that (12) holds with $\kappa = 2Dc(\overline{H})$.*

Proof. Suppose H is as in Theorem 3.1 and let ζ, ζ' be as in Theorem 3.2. For $i \in V(H)$, let $\gamma_i = \alpha_i \prod_{j=1}^{s-1} \beta_{i,\zeta_j}$, $\mu_i = \gamma_i \beta_{i,\zeta_s}$ and $\nu_i = \gamma_i \beta_{i,\zeta'_s}$, and set $\gamma = \sum \gamma_i$. Then $\mu_i, \nu_i \leq \gamma_i$ and (using the inequality $\prod \eta_i \geq 1 - \sum (1 - \eta_i)$ for $\eta_i \in [0,1]$) we have

$$\sum_i \gamma_i \geq \sum_i \alpha_i [1 - \sum_{j=1}^{s-1} (1 - \beta_{i,\zeta_j})] = 1 - \sum_{j=1}^{s-1} \sum_i \alpha_i (1 - \beta_{i,\zeta_j}) \geq 1 - (D-1)c(\overline{H}), \quad (13)$$

and

$$\sum_i \mu_i = \sum_i \gamma_i [1 - (1 - \beta_{i,\zeta_s})] \geq \gamma - \sum_i \alpha_i (1 - \beta_{i,\zeta_s}) \geq \gamma - c(\overline{H}), \quad (14)$$

and similarly $\sum \nu_i \geq \gamma - c(\overline{H})$; so Proposition 3.3 shows that (12) holds as claimed. $\square$

Proof of Theorem 3.1. Our approach here via (15) is similar to that of [1], which in turn was inspired by the "cavity" method of statistical physics; see e.g. [17].

Given an ordering $v_1, \ldots, v_n$ (with $n = |G|$) of $V(G)$, set $G_k = G - \{v_1, \ldots, v_k\}$. We have

$$t(G_k) = \sum_{\phi:V(G_k) \to H} W(\phi)$$

and may write

$$t(G_{k-1}) = \sum_{\phi:V(G_k) \to H} W(\phi) \sum_{i \in V(H)} \alpha_i \prod_{w \in N_{G_{k-1}}(v_k)} \beta_{i,\phi(w)}.$$

Thus

$$\frac{t(G_{k-1})}{t(G_k)} = \mathsf{E}_{G_k} \sum_{i \in V(H)} \alpha_i \prod_{w \in N_{G_{k-1}}(v_k)} \beta_{i,\phi(w)},$$

and

$$\ln t(G) = \sum_{k=1}^n \ln \mathsf{E}_{G_k} \sum_{i \in V(H)} \alpha_i \prod_{w \in N_{G_{k-1}}(v_k)} \beta_{i,\phi(w)}. \quad (15)$$

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We will use Theorem 3.2 to say that for large r the expectation in (15) is nearly determined by the r-neighborhood of v_k in G_{k-1}. To say this properly set, for a graph K and $v \in V(K)$,

$$\Psi_K(v) = \mathsf{E}_{K-v} \sum_{i \in V(H)} \alpha_i \prod_{w \in N_K(v)} \beta_{i,\phi(w)}. \tag{16}$$

We note right away that

$$\frac{1}{2} < \sum_{i \in V(H)} \alpha_i \prod_{w \in N_K(v)} \beta_{i,\phi(w)} \leq 1, \qquad \text{and so} \qquad \frac{1}{2} < \Psi_K(v) \leq 1. \tag{17}$$

The upper bound is trivial, while the lower bound follows from a computation similar to (13):

$$\sum_{i \in V(H)} \alpha_i \prod_{w \in N_K(v)} \beta_{i,\phi(w)} \geq \sum_{i \in V(H)} \alpha_i \Big(1 - \sum_{w \in N_K(v)} (1 - \beta_{i,\phi(w)}) \Big)$$

$$= 1 - \sum_{w \in N_K(v)} \sum_{i \in V(H)} \alpha_i (1 - \beta_{i,\phi(w)}) \geq 1 - Dc(\overline{H}) > \frac{1}{2}.$$

The assertion is then that for K of maximum degree at most D, $\Psi_K(v)$ is determined to within $o_r(1)$ by (the isomorphism type of) $B_K(v, r)$ (where $o_r(1) \to 0$ as $r \to \infty$); that is:

Lemma 3.5 *For any K, K' of maximum degree at most D, $v \in V(K)$ and $v' \in V(K')$ with $B_{K'}(v', r) \cong B_K(v, r)$, we have*

(a) $|\Psi_K(v) - \Psi_{K'}(v')| < D\kappa^r$,

(b) $|\ln \Psi_K(v) - \ln \Psi_{K'}(v')| < 2D\kappa^r$.

Proof. (a) The sum in (16) is a function of the multiset $M(v, \phi) = \{\phi(w) : w \in N_K(v)\}$. By Theorem 3.2 there is a coupling $(\tilde{\phi}, \tilde{\phi}')$ of ϕ and ϕ' chosen according to Pr_{K-v} and $\mathrm{Pr}_{K'-v'}$ so that $\mathrm{Pr}(M(v, \tilde{\phi}) \neq M(v', \tilde{\phi}')) \leq |N_K(v)|\kappa^r$. With this coupling, using the upper bound in (17),

$$|\Psi_K(v) - \Psi_{K'}(v')| \leq \mathrm{Pr}(M(v, \tilde{\phi}) \neq M(v', \tilde{\phi}')) \leq D\kappa^r.$$

(b) This is implied by (a) once we observe that $\Psi_K(v)$ is bounded below by (17). $\qquad\square$

Returning to the proof of Theorem 3.1, it's convenient to speak of an ordering σ of $V(G)$, thought of as a bijection from $V(G)$ to $[n]$ (again with $n = |V(G)|$). For such a σ and $v \in V(G)$, set $G(v, \sigma) = G[\{w \in V(G) : \sigma(w) \geq \sigma(v)\}]$. Then with σ a random (uniform) permutation of $V(G)$, (15) gives

$$\ln t(G) = \sum_{v \in V(G)} \mathsf{E}_\sigma \ln \Psi_{G(v,\sigma)}(v). \tag{18}$$

By Lemma 3.5 the contribution of v to (18) is determined up to $o_r(1)$ by $B_G(v, r)$. Precisely, let $U = B_G(v, r)$ and $U_\sigma = B_{G(v,\sigma)}(v, r)$, then

$$\mathsf{E}_\sigma \ln \Psi_{G(v,\sigma)}(v) = X_U + R, \tag{19}$$

where $X_U = \mathsf{E}_\sigma \ln \Psi_{U_\sigma}(v)$ depends on the ball $U = B_G(v, r)$ only, and $|R| < 2D\kappa^r$. By (17), we have $|X_U| < 1$.

Thus $|G|^{-1} \ln t(G) = \sum \mu(G, U)X_U + o_r(1)$ (with the sum over r-balls U) and

$$\left| |G_m|^{-1} \ln t(G_m) - |G_n|^{-1} \ln t(G_n) \right| < \sum |\mu(G_m, U) - \mu(G_n, U)| + o_r(1). \tag{20}$$

Finally, the right hand side of (20) can be made as small as desired by choosing a sufficiently large r and then m, n large enough to make the sum small. $\qquad\square$

Remark 1 Of course the condition on H in Theorem 3.1 can be replaced by any assumption that supports the conclusions of Lemma 3.5 (with some $o_r(1)$ in place of the explicit bounds given there). One notable example involves the *hard-core model*, in which $V(H) = \{0, 1\}$ and the weights are $\alpha_0 = 1/(1+\lambda)$, $\alpha_1 = \lambda/(1+\lambda)$, $\beta_{0,1} = \beta_{0,0} = 1$ and $\beta_{1,1} = 0$. Here the present results combined with [25] give the convergence in Theorem 3.1 provided $\lambda \le (D-1)^{D-1}/(D-2)^D \approx e/D$ (whereas Theorem 3.2 gives this for $\lambda < 1/D$).

Another very interesting example is that of counting q-colorings; thus H is the complete graph on $[q]$ (without weights, though to put it in the above framework we should replace $\alpha_i = 1$ by $\alpha_i = 1/q \ \forall i$). Here Theorem 3.2 gives convergence for $q > 2D$, but it seems reasonable to expect that $q \ge D + 1$ suffices. That this is at least true for large girth (that is, if we add the requirement that the girth of G_n tends to infinity), follows from the present arguments with Theorem 3.2 replaced by a result of Jonasson [11] which says (informally) that for a uniform q-coloring of an r-branching tree with $q \ge r+2$, the color of the root becomes nearly independent of the colors of the leaves as the depth of the tree grows. (We actually need this for trees in which each internal node has *at most* r children, but this version is easily seen to follow from the original.)

If we assume, in addition to large girth, that the G_n are *D-regular*, then we have (again for $q \ge D + 1$) the explicit limit

$$\frac{\ln \hom(G_n, H)}{|G_n|} \to \ln q + \frac{D}{2} \ln(1 - \frac{1}{q}). \tag{21}$$

This is one of the main results of [1], obtained there by combining the cavity method with a "rewiring" device (another idea from statistical physics [17]), used to maintain regularity. Here we have the result more easily: it follows from the observation that Johansson's theorem (which is also needed in [1]) implies that the expectations in (18) tend to $(D/2)\ln(1 - 1/q)$ as the girth grows; namely, it implies that for each $i \in [q]$ the events $\{\sigma(w) > \sigma(v)$ and $\phi(w) = i\}$ $(w \in N(v))$ are, for large girth, nearly independent, each with probability about $1/q$. (The key difference between the present argument and that of [1] is the use of the random ordering σ.)

Remark 2 An argument similar to the one above gives (for a left-convergent sequence $\{G_n\}$) convergence of $\{|G_n|^{-1}\mathsf{H}(\phi_{G_n})\}$, where H is (say binary) entropy and $\phi_{G_n} : V(G_n) \to H$ is chosen according to Pr_{G_n}. Here we should replace (15) by the "chain rule" expansion $\mathsf{H}(\phi) = \sum_v \mathsf{H}(\phi(v)|(\phi(w) : \sigma(w) > \sigma(v)))$. Getting to the analogue of (19) now requires an extra step: we should choose r_1 so that for any σ the law of $\phi(v)$ given $(\phi(w) : \sigma(w) > \sigma(v))$ is "nearly

determined" by $(\phi(w) : \sigma(w) > \sigma(v), w \in B(v, r_1))$, and then $r \ (> r_1)$ so that the law of the latter vector is nearly unaffected by the values taken by ϕ outside $B(v, r)$.

3.2 Proof via Mayer expansion

Our second proof relies on techniques which are well know in the mathematical statistical physics literature. To apply these techniques, we express $t(G, H)$ as the partition function of a so called abstract polymer system, express its logarithm in terms of an infinite series whose terms can be written down explicitly, and finally prove that for $c(\overline{H}) < 1/(8D)$, the series for $\frac{1}{|G|} \ln t(G, H)$ is absolutely convergent uniformly in $|G|$. This will allow us to take the limit in Theorem 3.1 term by term.

3.2.1 Stable sets, Mayer expansion, and Dobrushin's lemma

We start with some preliminaries from mathematical physics, reformulated here in a more combinatorial language. Let G be a graph and let $\mathcal{I}(G)$ denote the set of stable (independent) subsets of $V(G)$. We assign a variable x_i to each node i, and define the *multivariate stable set polynomial* as

$$\mathsf{stab}(G, \mathbf{x}) = \sum_{S \in \mathcal{I}(G)} \prod_{i \in S} x_i.$$

Note that $\mathsf{stab}(G, 1, \dots, 1) = \hom(G, H)$, where H is the graph on two adjacent nodes, with a loop at one of them (all weights being 1).

In the language of mathematical physics, the pair $(G, \mathbf{x})$ is called an *abstract polymer system*, and $\mathsf{stab}(G, \mathbf{x})$ is called the *partition function* of the abstract polymer system $(G, \mathbf{x})$ (see, e.g., [20], where the notion of an abstract polymer system was first introduced). Here we will be interested in the Taylor expansion of $\ln \mathsf{stab}(G, \mathbf{x})$ about $\mathbf{x} = (0, \dots, 0)$, known under the name of Mayer expansion in statistical physics. In a slightly less general context than the one considered here, this expansion was first derived in Malyshev [16], who in turn relied heavily on the work of Rota [18]. In the general context of an abstract polymer system, it goes back to [20].

For a sequence $v \in V^m$ of nodes of a simple graph G, let $G[v]$ denote the graph on $[m]$ in which i and j are adjacent if and only if v_i and v_j are equal or adjacent in G. (Note that v may contain repetitions.) The following lemma is a reformulation of a result of Seiler [20].

Lemma 3.6 *Let $G = (V, E)$ be a simple graph. For every $\mathbf{x} \in \mathbb{R}^V$ such that the series below is absolutely convergent, we have*

$$\ln \mathsf{stab}(G, \mathbf{x}) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{v \in V^m} \mathsf{cr}(G[v]) \prod_{i=1}^{m} x_{v_i}, \tag{22}$$

To prove absolute convergence of the expansion in (22), we use the following lemma which goes back to Dobrushin. In the form stated here, it can be found, e.g., in [4].

Lemma 3.7 *Let $G = (V, E)$ be a simple graph, and let $\mathbf{x} \in \mathbb{R}^V$ and $\mathbf{b} \in [0, \infty)^V$ be such*

$$\sum_{\substack{j \in V \\ ij \in E \text{ or } j=i}} \ln\left(1 + |x_j|e^{b_j}\right) \le b_i. \tag{23}$$

for all $i \in V$. Then the series in (22) is absolutely convergent, and

$$\left|\ln \mathsf{stab}(G, \mathbf{x})\right| \le \sum_{i \in V} \ln(1 + |x_i|e^{b_i}). \tag{24}$$

3.2.2 Mayer expansion for $\ln t(G, H)$

We can rewrite $\hom(G, H)$ in terms of the intersection graph $\mathcal{G} = L(\mathrm{CInd}(G))$ of connected induced subgraphs.

Lemma 3.8 *For every simple graph G and weighted graph H, define a vector $z \in \mathbb{R}^{\mathrm{CInd}(G)}$ by $z_F = z(F, \overline{H})$. Then $t(G, H) = \mathsf{stab}(\mathcal{G}, z)$.*

Proof.   By easy computation,

$$t(G, H) = \sum_{E' \subseteq E} (-1)^{|E'|} t(G', \overline{H}), \tag{25}$$

where $G' = (V(G), E')$. Using that $t(G', \overline{H})$ is multiplicative over the components of G' and that singleton components give a factor of 1, we get

$$t(G, H) = \sum_{E' \subseteq E} \prod_F (-1)^{|E(F)|} t(F, \overline{H}), \tag{26}$$

where the product extends over all non-singleton components of G'. Collecting terms that induce the same partition, we get

$$\begin{aligned}
t(G, H) &= \sum_{\mathcal{P} \in \Pi(V)} \prod_{Y \in \mathcal{P}} \sum_{F \in \mathrm{CSpan}(G[Y])} (-1)^{|E(F)|} t(F, \overline{H}) \\
&= \sum_{\mathcal{P} \in \Pi(V)} \prod_{Y \in \mathcal{P}} z(G[Y], \overline{H}) = \mathsf{stab}(\mathcal{G}, z).
\end{aligned}$$

$\square$

For any multiset $\{F_1, \dots, F_k\}$ of subgraphs, let $L(F_1, \dots, F_k)$ denote the intersection graph of $V(F_1), \dots, V(F_k)$. Combining Lemma 3.8 and Lemma 3.6, we get the following formula.

Corollary 3.9 *Let G be a simple graph and H, a weighted graph. If the series below is absolute convergent, we have*

$$\ln t(G, H) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{F_1, \dots, F_m \in \mathrm{CInd}(G)} \mathsf{cr}(L(F_1, \dots, F_m)) \prod_{j=1}^{m} z(F_j, \overline{H}). \tag{27}$$

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Next we establish the convergence condition (23) for $\mathbf{b}$ of the form $b_F = b|F|$. For vectors $\mathbf{b}$ of this form, it is clearly enough to prove that for all $i \in V$, we have

$$\sum_{\substack{F \in \mathrm{CInd}(G): \\ i \in V(F)}} \ln\left(1 + |z_F|e^{b|F|}\right) \leq b. \tag{28}$$

We in fact prove a stronger inequality. To state our result, we define

$$K = \frac{b + e^b}{\ln(1 + be^{-b})}, \qquad \epsilon = -\ln(DKc(\overline{H})) \qquad \text{and} \qquad \tilde{z}_F = e^{\epsilon(|F|-1)}z_F. \tag{29}$$

We will assume that $c(H) < 1/(DK)$, so that $\varepsilon > 0$.

In the special case of colorings, i.e., the case where H is the complete graph without loops, the next lemma was already shown in [4].

Lemma 3.10 *For every simple graph G with maximum degree D, every weighted graph H with edge weights in $[0,1]$, and every node $i \in V(G)$, we have*

$$\sum_{\substack{F \in \mathrm{CInd}(G) \\ V(F) \ni i}} |\tilde{z}_F|e^{b|F|} \leq b. \tag{30}$$

Remark 3 The lemma clearly implies condition (28). In fact, (28) holds even if z is replaced by $\tilde{z}$.

Proof. Using the bound in Lemma 2.4, it is enough to show that

$$\sum_{\substack{W \subseteq V: \\ i \in W, \ |W| \geq 2}} \mathsf{tree}(G[W])\lambda^{|W|-1} \leq be^{-b}. \tag{31}$$

where $\lambda = e^b/(KD)$. Consider a tree T contributing to $\mathsf{tree}(G[W])$. After removing the point i from T, the tree T decomposes into a certain number of connected components $T_1, \ldots, T_k$, with vertex sets $U_1, \ldots, U_k$. Note that $\Pi\{U_1, \ldots, U_k\}$ is a partition of $W \setminus \{i\}$ into disjoint subsets. Classifying the spanning trees of $G[W]$ according to these partitions, one easily obtains the identity

$$\mathsf{tree}(G[W]) = \sum_{\Pi} \prod_{U \in \Pi} \left(\mathsf{tree}(G[U]) \sum_{\substack{j \in U \\ ij \in E}} 1 \right), \tag{32}$$

where the sum runs over partitions of $W \setminus \{i\}$ into disjoint subsets. With the help of this identity, one easily bounds the left hand side of (31) by induction on the number of vertices in V. Indeed,

we first rewrite the left hand side as

$$\sum_{\substack{W \subseteq V: \\ i \in W, \; |W| \geq 2}} \mathsf{tree}(G[W])\lambda^{|W|-1}$$

$$= \sum_{\substack{W \subseteq V: \\ i \in W, \; |W| \geq 2}} \sum_{k=1}^{D} \frac{1}{k!} \sum_{\substack{U_1,\ldots,U_k \subseteq W\setminus\{i\} \\ W\setminus\{i\}=\bigcup_s U_s \\ U_s \cap U_r = \emptyset \text{ for } s \neq r}} \prod_{s=1}^{k} \left(\mathsf{tree}(G[U_s])\lambda^{|U_s|} \sum_{\substack{j \in U_s \\ ij \in E}} 1 \right)$$

$$= \sum_{k=1}^{D} \frac{1}{k!} \sum_{\substack{U_1,\ldots,U_k \subseteq V\setminus\{i\} \\ U_s \cap U_r = \emptyset \text{ for } s \neq r}} \prod_{s=1}^{k} \left(\mathsf{tree}(G[U_s])\lambda^{|U_s|} \sum_{\substack{j \in U_s \\ ij \in E}} 1 \right)$$ (33)

$$= \sum_{k=1}^{D} \frac{1}{k!} \sum_{\substack{j_1,\ldots,j_k \in N(i) \\ j_r \neq j_s \text{ for } s \neq r}} \sum_{\substack{U_1,\ldots,U_k \subseteq V\setminus\{i\} \\ U_s \cap U_r = \emptyset \text{ for } s \neq r \\ j_s \in U_s \text{ for all } s}} \prod_{s=1}^{k} \mathsf{tree}(G[U_s])\lambda^{|U_s|}.$$

Rewriting the first two sums as a sum over subsets of $N(i)$ and neglecting the non-overlap constraints on the sets U_s, we obtain the bound

$$\sum_{\substack{W \subseteq V: \\ i \in W, \; |W| \geq 2}} \mathsf{tree}(G[W])\lambda^{|W|-1} \leq \sum_{R \subseteq N(i)} \prod_{j \in R} \left(\sum_{\substack{U_j \subseteq V\setminus\{i\} \\ U_j \ni j}} \mathsf{tree}(G[U_j])\lambda^{|U_j|} \right)$$

$$= \sum_{R \subseteq N(i)} \prod_{j \in R} \left(\lambda + \lambda \sum_{\substack{U_j \subseteq V\setminus\{i\} \\ U_j \ni j \\ |U_j| \geq 2}} \mathsf{tree}(G[U_j])\lambda^{|U_j|-1} \right)$$ (34)

$$\leq \sum_{R \subseteq N(i)} \prod_{j \in R} \left(\lambda + \lambda b e^{-b} \right) = \left(1 + \lambda(1 + b e^{-b}) \right)^{|N(i)|} - 1$$

$$\leq e^{D\lambda(1+be^{-b})} - 1 = be^{-b}.$$

$\square$

The above lemma gives a bound on the tails

$$A_k = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{F_1,\ldots,F_m \in \mathrm{CInd}(G) \\ \sum(|F_i|-1) \geq k}} |\mathsf{cr}(L(F_1,\ldots,F_m))| \prod_{j=1}^{m} |z(F_j, \overline{H})|$$

of the expansion (27):

Lemma 3.11 *Let $b > 0$, let G be a graph with maximum degree D, and let H be a weighted graph with edgeweights in $[0,1]$. Then for every $k \geq 2$,*

$$A_k \leq be^{-\varepsilon k}|G|.$$

Proof. Bounding A_k by

$$A_k \leq e^{-\varepsilon k} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{F_1,\ldots,F_m \in \mathrm{CInd}(G) \\ \sum(|F_i|-1) \geq k}} |\mathsf{cr}(L(F_1,\ldots,F_m))| \prod_{j=1}^{m} |\tilde{z}_{F_j}|,$$

we can ignore the condition on $\sum(|F_i| - 1)$ to get

$$A_k \le e^{-\varepsilon k} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{F_1,\dots,F_m \in \mathrm{CInd}(G)} |\mathsf{cr}(L(F_1,\dots,F_m))| \prod_{j=1}^{m} |\tilde{z}_{F_j}| = e^{-\varepsilon k} \ln \mathsf{stab}(\mathcal{G}, \hat{\mathbf{z}}),$$

where $\hat{\mathbf{z}}$ denotes the vector $(-|\tilde{z}_F| : F \in \mathrm{CInd}(G))$. We use Theorem 3.7 and Lemma 3.10 to obtain the estimate

$$|A_k| \le e^{-\varepsilon k} \sum_{F \in \mathrm{CInd}(G)} \ln \left(1 + |\tilde{z}_F| e^{b|F|} \right) \le e^{-\varepsilon k} \sum_{i \in V} \sum_{\substack{F \in \mathrm{CInd}(G) \\ V(F) \ni i}} \ln \left(1 + |\tilde{z}_F| e^{b|F|} \right)$$

$$\le e^{-\varepsilon k} \sum_{i \in V} \sum_{\substack{F \in \mathrm{CInd}(G) \\ V(F) \ni i}} |\tilde{z}_F| e^{b|F|} \le e^{-\varepsilon k} |G| b.$$

$\square$

3.2.3 Proof of Theorem 3.1

We group the terms in Corollary 3.9 according to the subgraph of G induced by the union of the F_i. More precisely, for every graph F, define

$$v(F, H) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{F_1,\dots,F_m \in \mathrm{CInd}(F) \\ \cup_i V(F_i) = V(F)}} \mathsf{cr}(L(F_1,\dots,F_m)) \prod_{i=1}^{m} z(F_i, \overline{H}). \tag{35}$$

We note that $v(H, F) = 0$ if F is disconnected, since then $\mathsf{cr}(L(F_1,\dots,F_m)) = 0$. With this notation, we can also write (27) as

$$\ln t(G, H) = \sum_{F \in \mathrm{CInd}(G)} v(F, H) = \sum_F \mathrm{ind}_0(F, G) v(F, H), \tag{36}$$

where the last summation is extended over all isomorphism types of connected graphs F (clearly, graphs F with more than $|G|$ nodes contribute 0).

Hence

$$\frac{\ln t(G_n, H)}{|G_n|} = \sum_F \frac{\mathrm{ind}_0(F, G_n)}{|G_n|} v(F, H). \tag{37}$$

Here $\mathrm{ind}_0(F, G_n)/|G_n|$ tends to some value as $n \to \infty$ by left-convergence of the sequence (G_n). Hence

$$\left| \frac{\ln t(G_n, H)}{|G_n|} - \frac{\ln t(G_m, H)}{|G_m|} \right| \le \sum_{|F| \le k} \left| \frac{\mathrm{ind}_0(F, G_n)}{|G_n|} - \frac{\mathrm{ind}_0(F, G_m)}{|G_m|} \right| |v(F, H)|$$

$$+ \sum_{|F| > k} \left(\frac{\mathrm{ind}_0(F, G_n)}{|G_n|} + \frac{\mathrm{ind}_0(F, G_m)}{|G_m|} \right) |v(F, H)|$$

$$\le \sum_{|F| \le k} \left| \frac{\mathrm{ind}_0(F, G_n)}{|G_n|} - \frac{\mathrm{ind}_0(F, G_m)}{|G_m|} \right| |v(F, H)| + 2 A_k.$$

We can choose k large enough so that the last term is less than $\varepsilon/2$, and then the first term will be less than $\varepsilon/2$ if n and m are large enough.

This proves the theorem for $c(\bar{H}) < \frac{1}{KD}$. We choose b so as to minimize K. For $b = 2/5$ we get $K = 7.964 \cdots < 8$ (which is not far from the best we get by this method).

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4 Right convergence implies left convergence

4.1 Linear independence of homomorphism functions

The following lemmas extend some of the lemmas in [9].

Lemma 4.1 *Let $F_1, \ldots, F_k$ be non-isomorphic simple graphs. Then the matrices*

$$M_{\mathrm{inj}} = \left[\mathrm{inj}(F_i, F_j) \right]_{i,j=1}^{k}$$

and

$$M_{\mathrm{surj}} = \left[\mathrm{surj}(F_i, F_j) \right]_{i,j=1}^{k}$$

are nonsingular.

Proof. We may assume that the F_i are ordered so that for $i < j$, $|F_i| \leq |F_j|$ and $|E(F_i)| \leq |E(F_j)|$. Then the matrix M_{inj} is upper triangular and M_{surj} is lower triangular. Since both matrices have positive diagonal entries, they are nonsingular. □

Lemma 4.2 *Let $m \geq 1$ and let $\{F_1, \ldots, F_k\}$ be a finite family of non-isomorphic simple graphs closed under surjective homomorphic image. Then the matrix*

$$M_{\mathrm{hom}} = \left[\mathrm{hom}(F_i, F_j) \right]_{i,j=1}^{k}$$

is nonsingular.

Examples of such families are all simple graphs with at most q nodes, or all connected simple graphs with at most q nodes, or with at most m edges. We don't know if this proposition holds for more general (perhaps all?) families of graphs.

Proof. We can express homomorphisms by surjective and injective homomorphisms as follows:

$$\mathrm{hom}(F_i, F_j) = \sum_J \frac{\mathrm{surj}(F_i, J)\mathrm{inj}(J, F_j)}{\mathrm{aut}(J)},$$

where the summation extends over all simple graphs J for which $\mathrm{surj}(F_i, J) > 0$. All such graphs J belong to the family $\{F_1, \ldots, F_k\}$, which implies that if M_{inj} and M_{surj} are as in Lemma 4.1, and D_{aut} is the $k \times k$ diagonal matrix with the values $\mathrm{aut}(F_i)$ in the diagonal, then

$$M_{\mathrm{hom}} = M_{\mathrm{surj}} D_{\mathrm{aut}}^{-1} M_{\mathrm{inj}},$$

proving by Lemma 4.1 that M_{hom} is nonsingular. □

4.2 Convergence of graph sequences

Theorem 4.3 *Let $(G_1, G_2, \dots)$ be a sequence of simple graphs with degrees bounded by D, and assume that there is a $\delta > 0$ such that for every simple looped graph H with all degrees at least $(1 - \delta)|H|$, the sequence $\ln \hom(G_n, H)/|G_n|$ is convergent as $n \to \infty$. Then the sequence $(G_1, G_2, \dots)$ is left-convergent.*

Proof. Let $m \geq 1$ and let $\mathcal{F}_m = \{F_1, \dots, F_N\}$ be the set of all connected simple graphs with $2 \leq |F_i| \leq m$. By Lemma 4.2, the matrix

$$M = \Big[\hom(F_i, F_j) \Big]_{i,j=1}^N$$

is nonsingular.

Let $q > m$. Add $q - |F_i|$ isolated nodes to F_i and take the complement to get a simple graph H_i on $[q]$ with loops added at the nodes. We think of H_i as a weighted graph with all weights 1. Every node in H_i has degree at least $q - m$, so if we choose q large enough, the condition on H in the theorem is satisfied by every H_i.

Consider any graph G with all degrees at most D. We can rewrite (27) as follows:

$$\ln t(G, H_i) = \sum_F \mathrm{inj}_0(F, G) u(F, H_i), \tag{38}$$

where the summation extends over all connected graphs F, and

$$u(F, H_i) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{J_1, \dots, J_k \in \mathrm{Con}(F) \\ \cup J_i = F}} \mathsf{cr}(L(J_1, \dots, J_k)) \prod_{r=1}^{k} t(J_r, H_i - 1). \tag{39}$$

Here

$$t(J_r, H_i - 1) = q^{-|J_r|} (-1)^{|E(J_r)|} \hom(J_r, F_i),$$

and so

$$\prod_{r=1}^{k} t(J_r, H_i - 1) = (-1)^{\sum_r |E(J_r)|} q^{-\sum_r |J_r|} \prod_{r=1}^{k} \hom(J_r, F_i).$$

Note that the exponent of q is less than $-|F|$ except for $k = 1$ (when $J_1 = F$). Hence

$$u(F_j, H_i) = q^{-|F_j|} (-1)^{|E(F_j)|} (\hom(F_j, F_i) + O(q^{-1})), \tag{40}$$

for all $1 \leq i, j \leq N$ and

$$u(F, H_i) = O(q^{-|F|-1}) \tag{41}$$

if $|F| > m$. Here and in what follows, the constants implied in the O may depend on D and m (and so also on N), but not on q, G and ε.

By Lemma 4.2 it follows that if q is large enough, then the matrix $(u(F_i, H_j))_{i,j=1}^N$ is invertible. Furthermore, if $(w_{ij})_{i,j=1}^n$ denotes its inverse, then

$$w_{ij} = q^{|F_j|} (-1)^{|E(F_j)|} (M^{-1})_{ij} + O(q^{|F_j|-1}),$$

and so

$$|w_{ij}| = O(q^{|F_j|}) = O(q^m). \tag{42}$$

Write

$$\ln t(G, H_j) = \sum_{i=1}^{N} \mathrm{inj}_0(F_i, G) u(F_i, H_j) + R(G, H_j), \tag{43}$$

where

$$R(G, H_j) = \sum_{|F|>m} \mathrm{inj}_0(F, G) u(F, H_j) \tag{44}$$

is a remainder term, which we can estimate as follows, using Lemma 2.1(d):

$$|R(G, H_j)| \leq \sum_{r=m+1}^{\infty} \sum_{|F|=r} \mathrm{inj}_0(F, G) |u(F, H_j)| = \sum_{r=m+1}^{\infty} \sum_{|F|=r} \mathrm{inj}_0(F, G) O(q^{-r-1})$$

$$= \sum_{r=m+1}^{\infty} 2^{Dr} |G| O(q^{-r-1}) = O(q^{-m-2}) |G|. \tag{45}$$

We can view (43) as a system of N equations in the N unknowns $\mathrm{inj}_0(F_i, G)$, from which these unknowns can be expressed:

$$\mathrm{inj}_0(F, G) = \sum_{j=1}^{N} w_{ji} \ln t(G, H_j) + r_i(G), \tag{46}$$

where

$$r_i(G) = \sum_{j=1}^{N} w_{ji} R(G, H_j) = O(q^m) O(q^{-m-2}) |G| = O(q^{-2}) |G|.$$

Now let $\varepsilon > 0$ be given. Choosing q large enough, we have $|r_i(G)| < \varepsilon |G|$ for all $1 \leq i \leq N$ and every graph G. By hypothesis, if q is sufficiently large, then the sequence $\ln(t(G_n, H_j))/|G_n|$ will be convergent for every $1 \leq j \leq N$ as $n \to \infty$, and so we can choose a positive integer n_0 such that for $n, n' > n_0$, we have

$$\left| \frac{\ln t(G_n, H_j)}{|G_n|} - \frac{\ln t(G_{n'}, H_j)}{|G_n|} \right| \leq \varepsilon q^{-m}.$$

Then by (46),

$$\left| \frac{\mathrm{inj}_0(F_i, G_n)}{|G_n|} - \frac{\mathrm{inj}_0(F_i, G_{n'})}{|G_{n'}|} \right|$$

$$= \sum_{i=1}^{N} \left(w_{ji} \frac{\ln t(G_n, H_j)}{|G_n|} - w_{ji} \frac{\ln t(G_{n'}, H_j)}{|G_{n'}|} + \frac{r_i(G_n)}{|G_n|} - \frac{r_i(G_{n'})}{|G_{n'}|} \right)$$

$$\leq \sum_{i=1}^{N} |w_{ji}| \cdot \left| \frac{\ln t(G_n, H_j)}{|G_n|} - \frac{\ln t(G_{n'}, H_j)}{|G_{n'}|} \right| + O(\varepsilon)$$

$$\leq O(q^m) \varepsilon q^{-m} + O(\varepsilon) = O(\varepsilon).$$

This proves that the sequence $(\mathrm{inj}_0(F_i, G_n)/|G_n| : n = 1, 2, \dots)$ is convergent for all $F_i \in \mathcal{F}_m$. Since this holds for every $m \geq 1$, this proves the Theorem. $\square$

In the proof above, the graphs H_i have many twin nodes (all the added nodes). We can always replace these by a single node of large weight, to get a weighted graph H_i' on at most $m + 1$ nodes. The argument would remain essentially the same if we replaced the 0 edgeweights in H_i by $1 - \delta$ for any fixed $\delta > 0$. Hence we get the following variant:

Theorem 4.4 *Let $(G_1, G_2, \dots)$ be a sequence of simple graphs with degrees bounded by D, and let F be a simple graph. Assume that there is a $\delta > 0$ such that for every weighted graph H on $|F| + 1$ nodes with all edgeweights in $[1 - \delta, 1]$, the sequence $(\ln t(G_n, H))/|G_n|$ is convergent as $n \to \infty$. Then the sequence $\hom(F, G_n)/|G_n|$ is convergent.*

5 Food for thought

We mention some directions for further research.

Quantitative bounds. It would be interesting to make the relationship between the numbers $\hom(F, G)/|G|$ and $(\ln t(G, H))/|G|$ more explicit.

Limit objects. Benjamini and Schramm [2] associated a limit object with every left-convergent sequence of bounded degree graphs, in the form of a probability distribution (with some special properties) on countable rooted graphs with the same degree bound. Other constructions of limit objects include graphings [8] and measure preserving graphs [14]. The "left" quantities like $t(F, G)$ can be defined easily when G is replaced by such a limit object. Our Theorem 3.1 suggests that the quantities $\ln t(G, H)/|G|$ can also be extended. However, the definition (in other words, the description of the limiting value in terms of the limit object) is not clear at all.

Temperature. Most of the time we have considered weighted graphs H with $\alpha_H = 1$, whose edgeweights are between 0 and 1, and close to 1. Let us consider edgeweights of the form $\beta_{ij} = \exp(-B_{ij})$, where $B_{ij} \geq 0$, and normalize so that $\max_{i,j} B_{ij} = 1$. Also consider the weighted graph $H^{1/T}$, where T is a parameter which in statistical physics would be called the *temperature*, and the edge weights are raised to the $1/T$ power. In this notation

$$\hom(G, H^{1/T}) = \mathsf{E}_\phi \exp\Big(\frac{1}{T} \sum_{ij \in E(G)} B_{\phi(i)\phi(j)}\Big),$$

where ϕ is a random map $G \to H$. Furthermore,

$$c(\overline{H}) = \max_i \sum_j \alpha_j(1 - \beta_{ij}) \leq \max_i \frac{1}{T} \sum_j \alpha_j B_{ij} \leq \frac{1}{T}.$$

So it follows that if the temperature T is larger than $2D$ then for every left-convergent graph sequence (G_n), the partition functions $\hom(G_n, H^{1/T})$ are convergent.

What kind of convergence does it mean if the partition functions are convergent at smaller temperature as well? This is not a local property any more; still, is it related to some property "from the left"?

Distance. One would like to define a cut-distance type metric for bounded degree graphs. Let G_1 and G_2 be two graphs with degrees bounded by D on the same node set $V = [n]$. Let $e_i(S,T)$ denote the number of edges in G_i connecting S and T ($S, T \subseteq V$). Suppose that we have

$$|e_1(S,T) - e_2(S,T)| \leq \varepsilon n \tag{47}$$

for all $S, T \subseteq V$. Then for every weighted graph H with $V(H) = [q]$, $\alpha_H = 1$ and $1/B \leq \beta_{u,v} \leq B$ for some $B \geq 1$, we have

$$
\begin{aligned}
t(G_1, H) = \mathsf{E}_\phi \Big(\prod_{ij \in E(G_1)} \beta_{\phi(i)\phi(j)} \Big) &= \mathsf{E}_\phi \Big(\prod_{u,v \in V(H)} \beta_{uv}^{e_1(\phi^{-1}(u), \phi^{-1}(v))} \Big) \\
&\leq \mathsf{E}_\phi \Big(\prod_{u,v \in V(H)} \beta_{uv}^{e_2(\phi^{-1}(u), \phi^{-1}(v))} B^{\varepsilon n \binom{q}{2}} \Big) \\
&= B^{\varepsilon n \binom{q}{2}} t(G_2, H).
\end{aligned}
$$

Hence

$$\left| \frac{\ln t(G_1, H)}{n} - \frac{\ln t(G_2, H)}{n} \right| \leq \varepsilon \binom{q}{2} \ln B.$$

By Theorem 4.4, this implies that $\hom(F, G_1)/|G_1|$ and $\hom(F, G_2)/|G_2|$ are also close if n is large enough.

The trouble is that condition (47) is too strong: it does not hold for two random D-regular graphs, for example. Perhaps it is possible to replace it by some condition asserting that it holds "on the average"?

References

[1] A. Bandyopadhyay and D. Gamarnik, Counting without sampling. Asymptotics of the log-partition function for certain statistical physics models, *Random Structures & Algorithms* **33**, 2008.

[2] I. Benjamini and O. Schramm: Recurrence of Distributional Limits of Finite Planar Graphs, *Electronic J. Probab.* **6** (2001), paper no. 23, 1–13.

[3] I. Benjamini, O. Schramm, A. Shapira: Every Minor-Closed Property of Sparse Graphs is Testable, *40th Ann. ACM Symp. on Th. Comp.* (2008), 393–402.

[4] C. Borgs: Absence of Zeros for the Chromatic Polynomial on Bounded Degree Graphs, *Combinatorics, Probability and Computing* **15** (2006), 63-74.

[5] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent Graph Sequences I: Subgraph frequencies, metric properties, and testing, *Advances in Math.* **219** (2008), 1801–1851.

[6] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent Graph Sequences II: Multiway Cuts and Statistical Physics (submitted),

http://www.cs.elte.hu/~lovasz/ConvRight.pdf

[7] R.L. Dobrushin: Estimates of semi-invariants for the Ising model at low temperatures. In: Topics in Statistical and Theoretical Physics, Vol. 177 of American Mathematical Society Translations, Ser. 2 (1996), 59-81.

[8] G. Elek: On limits of finite graphs, *Combinatorica* **27** (2007), 503–507.

[9] P. Erdös, L. Lovász, J. Spencer: Strong independence of graphcopy functions, in: *Graph Theory and Related Topics*, Academic Press, 165–172.

[10] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter, Berlin, 1988.

[11] J. Jonasson, Uniqueness of uniform random colorings of regular trees, *Statistics and Probability Letters* **57** (2002), 243-248.

[12] D. Levin, Y. Peres and E. Wilmer, *Markov Chains and Mixing Times*, AMS, Providence, 2008.

[13] L. Lovász: *Combinatorial Problems and Exercises*, North-Holland Publishing Co., Amsterdam, 1993; AMS Chelsea Publishing, 2007.

[14] L. Lovász: Very large graphs, in: *Current Developments in Mathematics 2008* (eds. D. Jerison, B. Mazur, T. Mrowka, W. Schmid, R. Stanley, and S. T. Yau), International Press, Somerville, MA 2009, 67–128.

[15] R. Lyons: Asymptotic enumeration of spanning trees *Combin. Prob. Comput.* **14** (2005) 491–522.

[16] Malyshev, V.A.: Uniform Cluster Estimates for Lattice Models. Commun. Math. Phys. **64** (1979) 131–157.

[17] M. Mezard and G. Parisi, The cavity method at zero temperature, *J. Statist. Phys.* **111** (2003), 1-34.

[18] G.-C. Rota: *On the foundations of combinatorial theory, I Theory of Möbius functions.* Z. Wahrsch. Verw. Gebiete **2** (1964) 340-368.

[19] A.D.Scott and A.D.Sokal: On Dependency Graphs and the Lattice Gas, Combinatorics, Probability and Computing (2006) 15, 253–279.

[20] E. Seiler: *Gauge theories as a problem of constructive field theory and statistical mechanics.* Lecture notes in physics, Springer, Berlin-Heidelberg-New York, 1982.

[21] J.B. Shearer: On a problem of Spencer. Combinatorica 5 (1985), 241–245.

[22] O. Schramm: Hyperfinite graph limits,

http://arxiv.org/PS_cache/arxiv/pdf/0711/0711.3808v1.pdf

[23] A. Sokal: Bounds on the Complex Zeros of (Di)Chromatic Polynomials and Potts-Model Partition Functions, Combin. Probab. Comput. 10 41–77.

[24] R. Stanley: *Enumerative combinatorics*, Volume 2, Cambridge University Press, Cambridge, 1999.

[25] D. Weitz, Counting independent sets up to the tree threshold, pp. 140-149 in *Proc. 38th Annual ACM Symposium on Theory of Computing*, ACM, New York, 2006.

Appendix: Grids, a case study

Let $P_n \square P_m$ denote the $n \times m$ grid (the Cartesian product of a path with n nodes and a path with m nodes); let $C_n \square P_m$ denote the $n \times m$ cylindrical grid (the Cartesian product of a cycle with n nodes and a path with m nodes), and let $C_n \square C_m$ denote the $n \times m$ toroidal grid (the Cartesian product of a cycle with n nodes and a cycle with m nodes). The sequences of $n \times m$ grids, cylindrical grids and toroidal grids (and any merging of these three) are trivially left-convergent if $m, n \to \infty$.

Grids are also right-convergent in a strong sense:

Proposition 5.1 *For every weighted graph H, the sequence $\ln \hom(P_n \square P_m)/nm$ is convergent as $n, m \to \infty$.*

Proof. Let

$$s_0 = \inf_{n,m} \frac{\ln \hom(P_n \square P_m)}{nm}.$$

Fix an $\varepsilon > 0$, and choose $a, b \geq 1$ so that

$$\frac{\ln \hom(P_a \square P_b)}{ab} \leq s_0 + \varepsilon,$$

and write $n = au + r$ and $m = bv + s$ where $0 \leq r < a$ and $0 \leq s < b$. Using that trivially

$$\hom(P_{n_1+n_2} \square P_m, H) \leq \hom(P_{n_1} \square P_m, H)\hom(P_{n_2} \square P_m, H), \tag{48}$$

it follows by an argument which could be called "2-dimensional Fekete Lemma" that

$$\frac{\ln \hom(P_n \square P_m)}{nm} \leq \frac{u \ln \hom(P_a \square P_m) + \ln \hom(P_r \square P_m)}{nm}$$
$$\leq \frac{uv \ln \hom(P_a \square P_b) + u \ln \hom(P_a \square P_s) + v \ln \hom(P_r \square P_b) + \ln \hom(P_r \square P_s)}{nm}$$
$$\leq s_0 + \varepsilon + O(\frac{1}{n} + \frac{1}{m})$$

if $n, m \to \infty$. $\qquad\square$

The situation is more complicated with cylindrical grids:

Proposition 5.2 (a) *If n is restricted to even numbers, then for every weighted graph H, the sequence* $\ln\hom(C_n\Box P_m)/nm$ *is convergent as* $n,m\to\infty$.

(b) *If H is connected and nonbipartite, then the sequence* $\ln\hom(C_n\Box P_m)/nm$ *is convergent as* $n,m\to\infty$.

Proof. (a) We may assume that the edgeweights of H are in $[0,1]$. Then

$$\hom(C_n\Box P_m) \le \hom(P_n\Box P_m),$$

and hence

$$\limsup_{n,m\to\infty}\frac{\ln\hom(C_n\Box P_m)}{nm} \le \limsup_{n,m\to\infty}\frac{\ln\hom(P_n\Box P_m)}{nm} \le s_0.$$

On the other hand, consider the subsets of nodes $A_1 = \{(0,x) :\quad x = 1\ldots m\}$ and $A_1 = \{(n/2,x): \ x = 1\ldots m\}$ in $L'_{n,m}$. We think of $L'_{n,m}$ as two $(n/2+1)\times m$ grids B_1 and B_2, glued together along A_1 and A_2. For any fixed map $\sigma: \ A_1\cup A_2 \to V(H)$, let M_σ denote the number of H-colorings of B_1 extending σ. Then the number of H-colorings of $L'_{n,m}$ extending σ is M_σ^2, and so

$$\hom(C_n\Box P_m, H) = \sum_\sigma M_\sigma^2,$$

while

$$\hom(P_{n/2+1}\Box P_m, H) = \sum_\sigma M_\sigma.$$

Thus by Cauchy-Schwartz,

$$\hom(C_n\Box P_m, H) = \sum_\sigma M_\sigma^2 \ge \frac{1}{|V(H)|^{2m}}\left(\sum_\sigma M_\sigma\right)^2$$
$$= \frac{1}{|V(H)|^{2m}}\hom(P_{n/2+1}\Box P_m, H)^2,$$

and so

$$\hom(C_n\Box P_m, H) \ge \frac{1}{nm}\left(2\ln\hom(P_{n/2+1}\Box P_m, H) - 2m\ln|V(H)|\right)$$
$$= (1+\frac{2}{m})\frac{\ln\hom(P_{n/2+1}\Box P_m, H)}{(n/2+1)m} - \frac{2}{n}$$
$$\ge (1+\frac{2}{m})s_0 - \frac{2}{n}.$$

This lower bound tends to s_0 as $n,m\to\infty$.

(b) As before, it is trivial that

$$\limsup_{n,m\to\infty}\frac{\ln\hom(C_n\Box P_m)}{nm} \le s_0,$$

so our task is to estimate $\hom(C_n\Box P_m)$ from below.

Fix any $\varepsilon > 0$, and then fix an $n_0 > 0$ so that if $n, m \geq n_0$ then $\ln \hom(P_n \square P_m)/(nm) \geq s_0 - \varepsilon$.

For a given $m \geq n_0$, we construct an auxiliary weighted graph G as follows. The nodes of G are all maps $V(P_m) \to V(H)$ with positive weight, where the weight of a map ϕ is $\prod_{i \in V(P_m)} \alpha_{\phi(i)}(H) \prod_{ij \in E(P_m)} \beta_{\phi(i)\phi(j)}(H)$. Two maps ϕ, ψ are connected by an edge with weight $b_{\phi,\psi} = \prod_{i \in V(P_m)} \beta_{\phi(i)\psi(i)}$ if this weight is positive.

Claim. No connected component of G is bipartite.

Consider any node x of G; we want to show that there is a walk in G of odd length starting and ending at x.

The node x is a map of P_m into H, which can also be viewed as a walk W in H with m nodes. Since H is connected and nonbipartite, we can extend W to a closed walk $W' = (v_1, v_2, \ldots, v_p)$ of odd length p (so $W = (v_1, \ldots, v_m)$). Let $W_i = (v_i, v_{i+1}, \ldots, v_{i+m-1})$ (where the indices are taken modulo p). The W_i corresponds to a node in H and $(W_1, \ldots, W_p)$ is a walk in H of odd length containing W. This proves the Claim.

Now observe that

$$\hom(P_n \square P_m, H) = \hom(P_n, G)$$

and

$$\hom(C_n \square P_m, H) = \hom(C_n, G).$$

Since G is connected and nonbipartite, it is easy to show that $\ln \hom(P_n \square P_m)/(nm)$ and $\ln \hom(C_n \square P_m)/(nm)$ tend to the same value as $n \to \infty$, and we know that this value is at least $s_0 - \varepsilon$. So for a fixed $m \geq n_0$, if $n \geq n_0(m, \varepsilon)$, then $\ln \hom(P_n \square P_m)/(nm) \geq s_0 - 2\varepsilon$. Choose

$$m_0 = \max\left\{ n_0, \frac{2 \ln q}{\varepsilon} \right\},$$

and let

$$N_0 = \max\{ \frac{s_0}{\varepsilon} m_0, n_0(m_0, \varepsilon) \}.$$

Assume that $n, m \geq N_0$. Consider the cylindrical grid $C_n \square P_{m_0}$. Since $n \geq n_0(m_0, \varepsilon)$, this grid has at least $\exp((s_0 - 2\varepsilon)nm_0)$ H-colorings. Hence there is a way to fix the map on the two "boundary" cycles so that after fixing it, we still have at least

$$\frac{\exp((s_0 - 2\varepsilon)nm_0)}{q^{2n}} \geq \exp((s_0 - 3\varepsilon)nm_0)$$

extensions. Let α_1 and α_2 denote these maps of the two boundary cycles.

Now consider the cylindrical grid $C_n \square P_{m_0}$. Map the first, m_0-th, $(2m_0 - 1)$-st etc. n-cycles alternatingly according to α_1 and α_2. Then the number of ways to extend this to the layer between two such cycles is at least $\exp((s - 3\varepsilon)nm_0)$, so the number of ways to extend the mapping to the whole graph is at least

$$\exp\left((s_0 - 3\varepsilon)nm_0 \lfloor \frac{m}{m_0} \rfloor \right) \geq \exp\left((s_0 - 3\varepsilon)nm_0 (\frac{m}{m_0} - 1) \right) \geq \exp((s_0 - 4\varepsilon)nm).$$

Hence

$$\ln \hom(C_n \square P_m)/(nm) \geq s_0 - 4\varepsilon.$$

$\square$

For toroidal grids, the situation is similar (and so are the proofs), and we only state the result:

Proposition 5.3 (a) *If n and m are restricted to even numbers, then for every weighted graph H, the sequence $\ln \hom(C_n \square C_m)/(nm)$ is convergent as $n, m \to \infty$.*

(b) *If H is connected and nonbipartite, then the sequence $\ln \hom(C_n \square C_m)/(nm)$ is convergent as $n, m \to \infty$.* $\square$