First order logic on Galton-Watson trees

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The key parts

- Galton-Watson trees, Poisson thinning
- First order language on rooted trees
- Tools used: Ehrenfeucht games
- Our results on probabilities of first order sentences

The Galton-Watson branching process

- For this talk, offspring distribution Poisson(λ) for $\lambda > 0$.
- Start with a single vertex, called root (denoted R). Let it have X_0 children where $X_0 \sim \text{Poi}(\lambda)$.
- Conditioned on $X_0 = m$, i.e. R having m children $v_1, \ldots, v_m$, let v_i have X_i children, where $X_1, \ldots, X_m$ i.i.d. Poi(λ).
- Continue like this. Call the random tree generated T_λ.
- When $\lambda > 1$, T_λ survives with positive probability. When $\lambda \leq 1$, T_λ is finite almost surely.

Reason for using Poisson: thinning property

- The *Poisson thinning property* makes computations easier.
- Suppose an individual has $X \sim \text{Poi}(\lambda)$ children, where each child is independently classified into one of k categories with probabilities $p_i, 1 \leq i \leq k$.
- If X_i is the number of children in category i, then $X_1, \ldots, X_k$ are independent, and $X_i \sim \text{Poi}(\lambda p_i)$.

Broad picture of our research

- Mathematical logic provides various *classes of properties* on underlying abstract structures, such as graphs and more specifically, trees.
- Example – *first order properties* describe local, finite structures inside a rooted tree.
- Example – *monadic second order properties* describe more global structures inside a rooted tree.
- We shall focus on only first order properties in this talk.

Formal view of FO

First order language on rooted trees comprises finite sentences consisting of:

- Root R as a special vertex, other vertices denoted by $x, y, z \ldots$ etc.;
- Relations: equality ($x = y$) and parent-child ($\pi(y) = x$, which denotes that x is the parent of y);
- Boolean connectives $\vee$, $\wedge$, $\neg$, $\implies$, $\Leftrightarrow$ etc.;
- Quantifications: existential ($\exists$) and universal ($\forall$), allowed *only over vertices*.

Definition

The *quantifier depth* of an FO property is the minimum number of *nested quantifiers* required to express the property.

What typical, simplest FO look like: some examples

- Fixing a finite tree T_0, the property

$$A[T_0] = \{\exists \text{ a copy of } T_0 \text{ inside the tree}\}.$$

For example, the FO sentence $\{\exists$ a vertex with exactly 1 child$\}$, which can be written as

$$B = \Big\{\exists x \exists y \big[\, \{\pi(y) = x\} \wedge \{\forall z \, [\pi(z) = x \implies z = y]\} \,\big]\Big\}.$$

- Fixing a finite tree T_0 of depth d, the property

$$B[T_0] = \{\text{radius-}d \text{ neighbourhood of } R \cong T_0\}.$$

For example, the FO sentence $A = \{R$ has at least 2 children$\}$, which can be written as

$$A = \{\exists x_1 \exists x_2 \, [\{\pi(x_1) = R\} \wedge \{\pi(x_2) = R\}] \, \}.$$

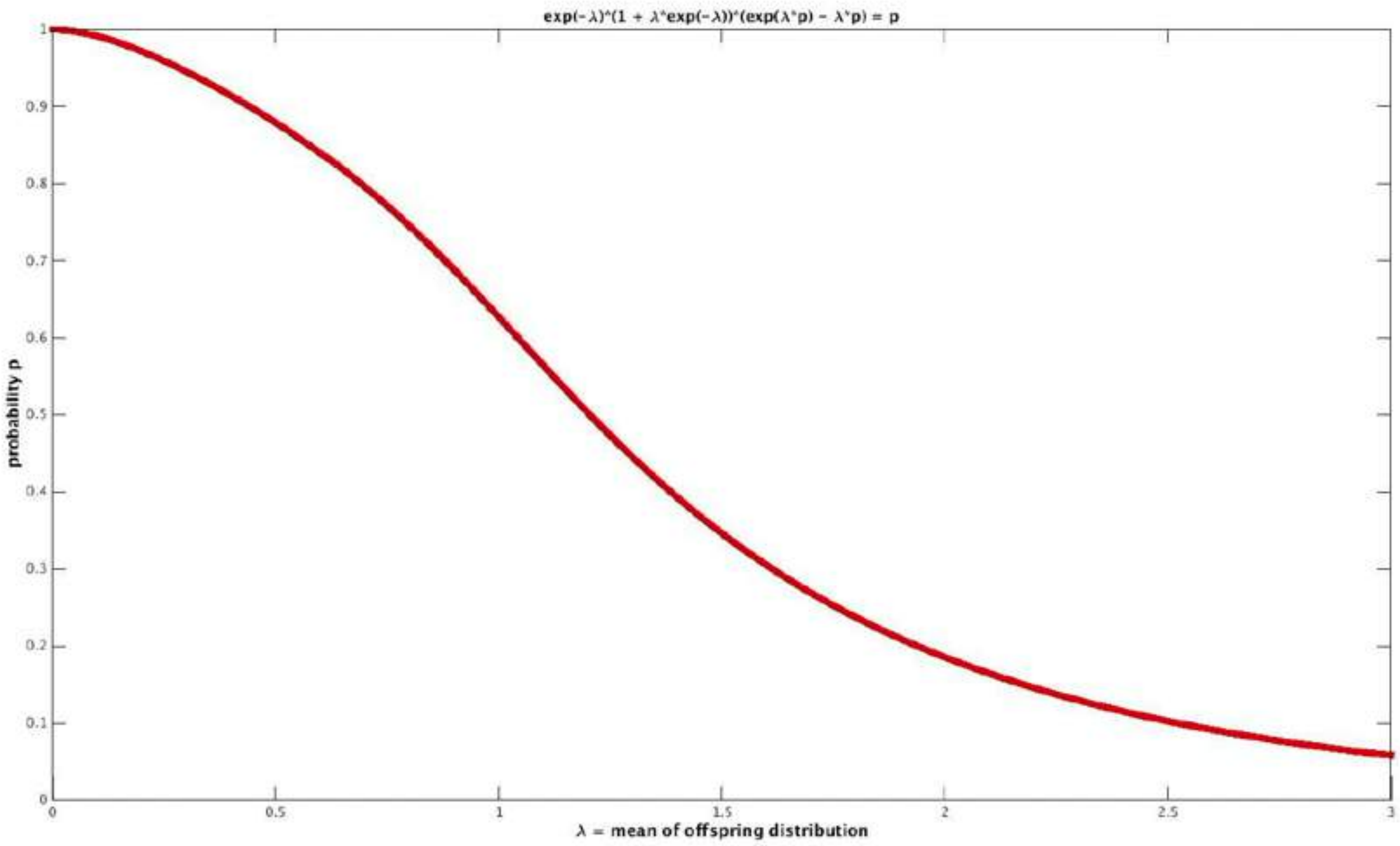
$A = \exists$ node with exactly one child and one grandchild

- **Finite** state space $\Sigma = \{\bullet, \bullet, \bullet\}$.
- $\bullet$: A holds; $\bullet$: root has one child, $\neg A$ holds; $\bullet$: all else.
- Parent vertex colour determined by counts of children of each colour.
- $(\geq 1, -, -) \Rightarrow \bullet$
  $(0, 1, 0) \Rightarrow \bullet$
  $(0, 0, 1) \Rightarrow \bullet$
- $x = \Pr[\bullet]$, $y = \Pr[\bullet]$, $z = \Pr[\bullet]$. Equations using Poisson thinning:

$$x = 1 - e^{-x\lambda} + y\lambda e^{-\lambda}, \quad y = z\lambda e^{-\lambda}.$$

- Solution $x = f_A(\lambda)$ **unique, smooth function of** λ.

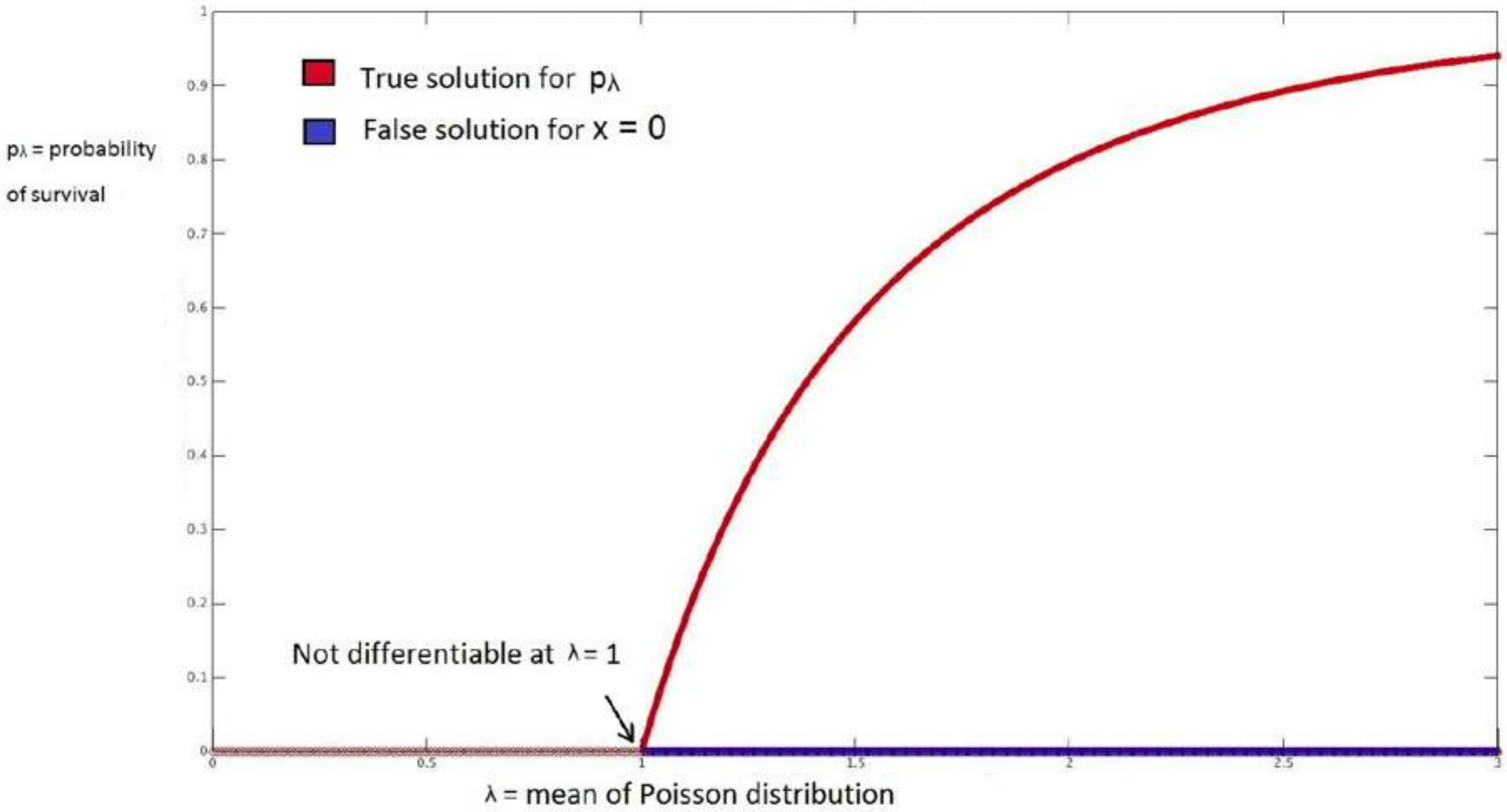
$1 - f_A(\lambda)$, as a function of λ



In contrast, a non-FO property

- B = the tree survives. This is an existential monadic second order property, cannot be expressed as an FO sentence.
- $\bullet$: B holds; $\bullet$: B does not hold.
- Parent vertex colour determined by counts of children of each colour.
  - $(\geq 1, -) \Rightarrow \bullet$.
  - $(0, -) \Rightarrow \bullet$.
- $x = \Pr[\bullet]$, $x = 1 - e^{-x\lambda}$.
- Solution $x = f_B(\lambda)$ **not unique** when $\lambda > 1$, not differentiable at $\lambda = 1$.

Probability of immortality, as a function of λ



Our aim

- To come up with a general method for analyzing *any* given FO sentence, not just specific examples.

For this, we need *Ehrenfeucht games*, the tool that serves as a bridge between mathematical logical and structural descriptions of logical properties on graphs (in our case, rooted trees).

Ehrenfeucht game for FO

- Played on trees T_1 with root R_1 and T_2 with root R_2, for k rounds, between players *Spoiler* and *Duplicator*.
- Each round consists of a move by Spoiler, followed by a move by Duplicator. Spoiler selects a vertex from *either* of T_1 and T_2; Duplicator then selects a vertex from the *other* tree.
- Suppose x_i is selected from T_1 and y_i from T_2 in round i. Set $x_0 = R_1$ and $y_0 = R_2$ as roots are special vertices.
- Duplicator wins if for all $0 \leq i, j \leq k$:
  - $x_i = x_j$ iff $y_i = y_j$;
  - x_i is the parent of x_j iff y_i is the parent of y_j.
- Denote this game by $\text{EHR}[T_1, T_2, k]$.

Connection of Ehrenfeucht games with FO properties

- Define $T_1 \sim_k T_2$ if Duplicator wins $\text{EHR}[T_1, T_1, k]$.
- $\sim_k$ is an equivalence relation on the space of rooted trees.
- Let Σ_k be the set of all equivalence classes under $\sim_k$.
- Crucially, Σ_k is *finite*.

Theorem (Well-known)

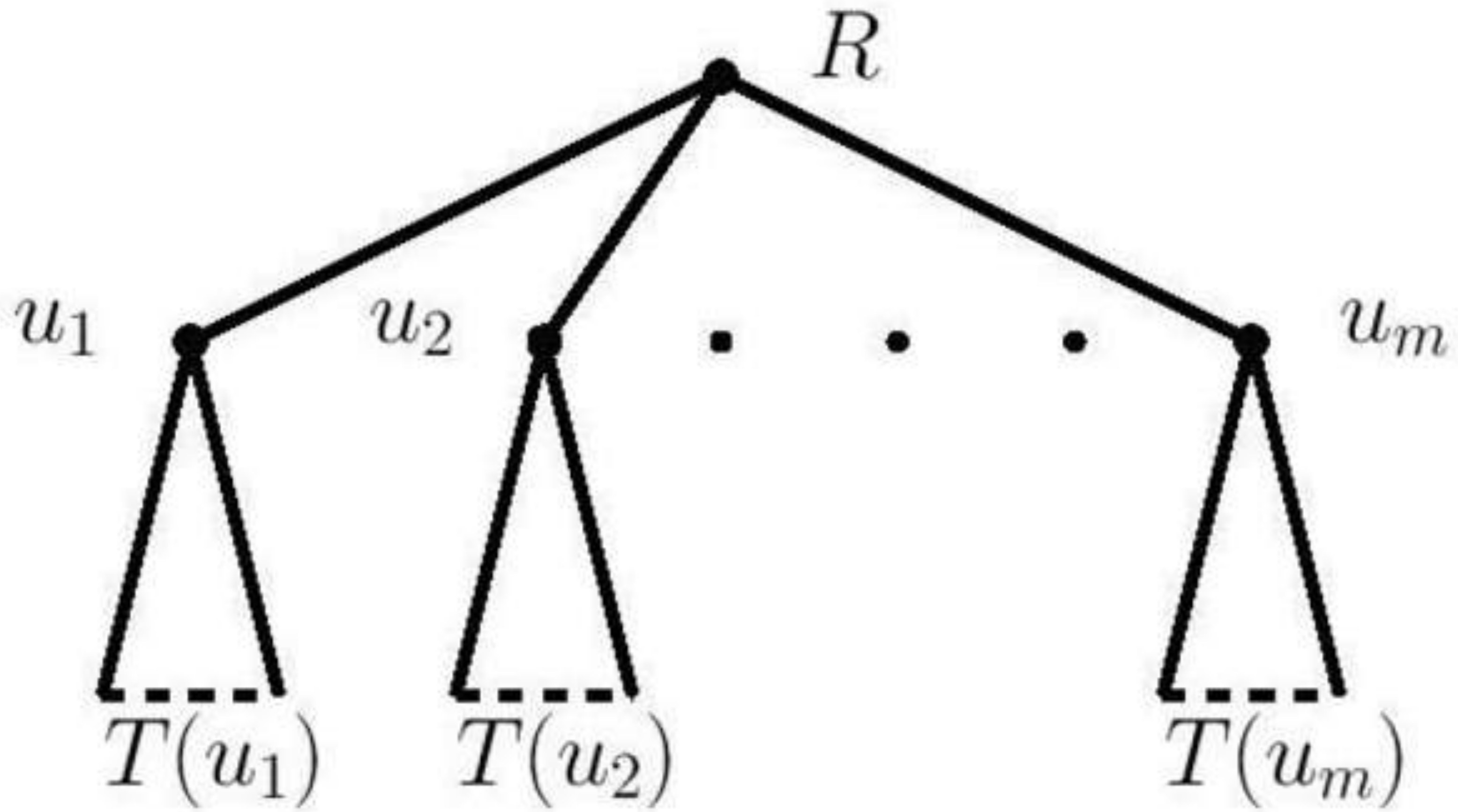
If $T_1 \sim_k T_2$, then for any FO property A of quantifier depth at most k, either both T_1 and T_2 satisfy A, or both T_1 and T_2 satisfy $\neg A$.

Thus, it is enough for us to analyze the probabilities of all $\sigma \in \Sigma_k$.

Probabilities of FO properties

- Let P_λ the measure induced by Poi(λ) Galton-Watson branching process.
- Fix $k \in \mathbb{N}$. We are interested in $P_\lambda[A]$ for any given FO property A of quantifier depth k.
- Set $\Sigma_k(A)$ to be the set of all $\sigma \in \Sigma_k$ such that A holds for each tree in σ. Then
$$P_\lambda[A] = \sum_{\sigma \in \Sigma_k(A)} P_\lambda[\sigma].$$
- Hence enough to analyze $P_\lambda[\sigma]$ for all $\sigma \in \Sigma_k$.

Recursion relation for the equivalence classes



- For $\sigma \in \Sigma_k$, let $n_\sigma = \#\{u_i : T(u_i) \in \sigma\}$.
- Define

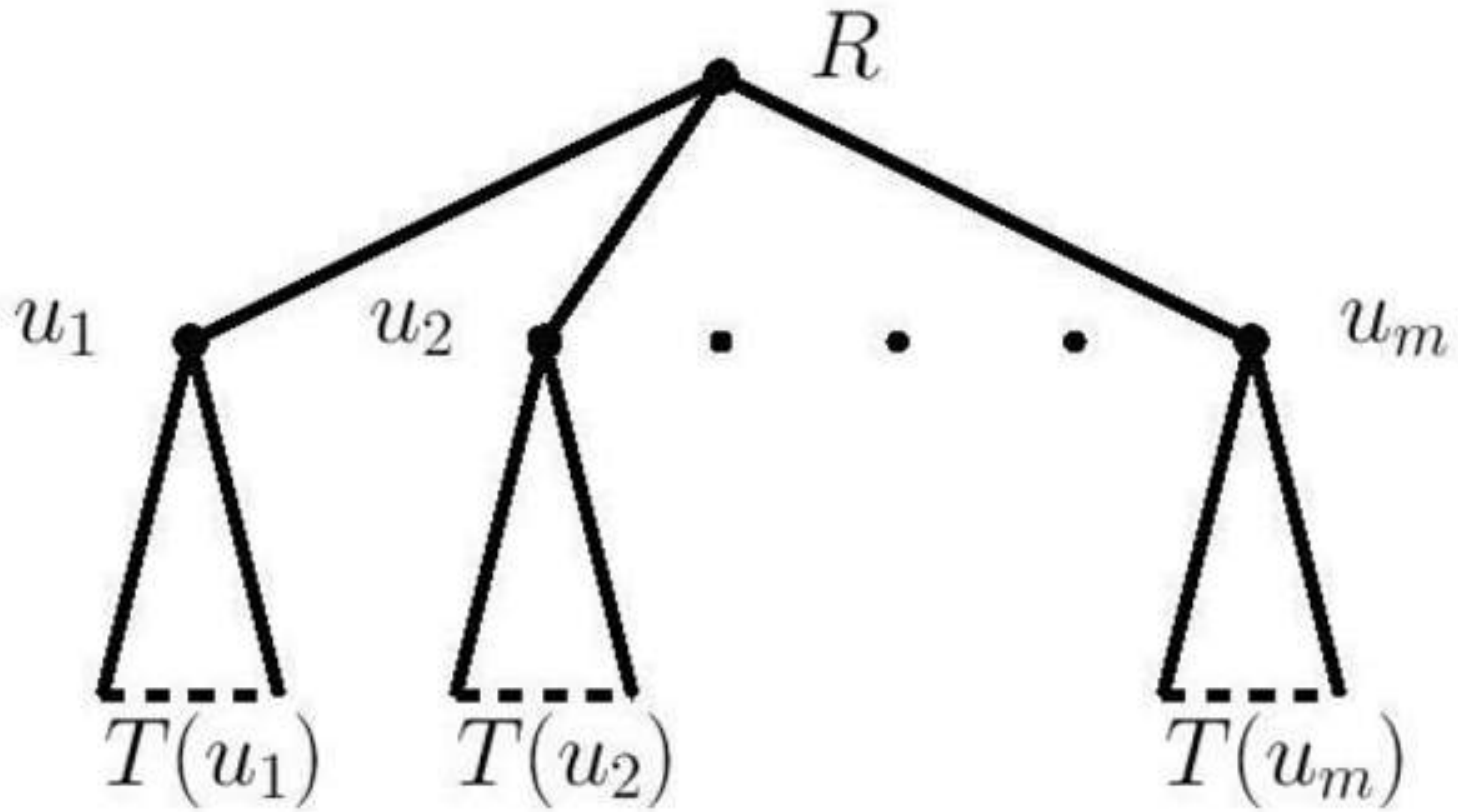
$$\vec{n} = \left(\min\{n_\sigma, k\} : \sigma \in \Sigma_k \right).$$

This completely determines the equivalence class to which the entire tree belongs.

A natural iteration from the recursion

1. From previous slide, we get a rule $\Gamma : \{0, 1, \ldots k\}^{\Sigma_k} \to \Sigma_k$, such that $\Gamma(\vec{n})$ is the equivalence class containing the entire tree.
2. Let $\mathcal{D}$ be the set of all probability distributions on Σ_k. Define **distributional map** $\Psi : \mathcal{D} \to \mathcal{D}$ as follows.
3. Choose $\vec{x} \in \mathcal{D}$. Let root R have Poi(λ) children.
4. To each child of R, mutually independently, assign a value from Σ_k according to distribution $\vec{x}$. This gives us a random vector $\vec{n}$.
5. Get the (random) **induced equivalence class** of the entire tree using the rule Γ.
6. We define $\Psi(\vec{x})$ to be the distribution of the induced equivalence class of the entire tree.

Recursion relation for the equivalence classes



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Distributional map and fixed points

Observation

If we define

$$\vec{p}(\lambda) = \big(P_\lambda(\sigma) : \sigma \in \Sigma_k\big)$$

then $\vec{p}(\lambda)$ *is a fixed point of* Ψ.

Theorem (P., Spencer)

- Ψ *is a contraction.*
- $\vec{p}(\lambda)$ *is the unique fixed point of* Ψ.
- $\vec{p}(\lambda)$ *is analytic in* λ.

FO probabilities conditioned on survival

Consider only $\lambda > 1$. Recall, for any finite tree T_0,

$$A[T_0] = \{\exists \text{ copy of } T_0 \text{ inside } T_\lambda\}.$$

Theorem (P., Spencer)

- *Conditioned on T_λ's survival, for every T_0, $A[T_0]$ holds P_λ-almost surely.*
- *Barring a bad set of infinite trees of P_λ-measure 0, the FO properties of quantifier depth k that hold for a typical infinite tree are completely captured by the neighbourhood of the root of radius $\approx 3^{k+2}$.*
- *The probabilities conditioned on survival are nice functions of λ and p_λ (the survival probability of T_λ) – compositions of polynomials and iterated exponentiations.*

Conclusion and further work

- Previous results give us a complete description of probabilities of FO properties on rooted T_λ.
- We have further explored existential monadic second order properties (EMSO's) on rooted Galton-Watson (GW) trees (and even more generally, rooted random trees with some restrictions on the probability measures). This is joint work with Holroyd, Levy and Spencer.
- We have also explored tree automata and their probabilities on the Galton-Watson branching process. These are closely related to EMSO's. This is joint work with Johnson and Skerman.

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Thank you!