Stochastic explosions in branching processes and non-uniqueness for nonlinear PDE

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NSE - quick look

Incompressible Navier-Stokes Equations:

$$\left\| \begin{array}{c} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = 0 \\ \nabla \cdot u = 0 \\ u(x,0) = u_0(x) \ \ (\text{\& B.C.} \dots) \end{array} \right.$$

Formally:

$$\left\| \begin{array}{c} u_t + Au + B(u,u) = 0 \\ u(0) = u_0 \end{array} \right.$$

Known: via Approximations/Fixed Point arguments:

- local existence and uniqueness of *smooth* solutions;
- global existence and uniqueness for *small initial data*;
- global existence of weak (Leray-Hopf) solutions; uniqueness?

Open Problems:

- Global well-posedness: existence and uniqueness...
- Regularity problem: Can smooth solutions blow up?

Scaling and Regularity

$$u(x,t) \ - \ \text{soln.} \quad \Rightarrow \quad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \text{ is also a soln.}$$

What is known globally:
"$\|u(t)\|^2 \leq \|u_0\|^2 < \infty$" – not enough for regularity.

The most basic regularity criterion:
"$\|A^{\frac{1}{2}} u(t)\|^2 < \infty \quad \Rightarrow$ regularity."

Basic small initial data result:
"$\|A^{\frac{1}{2}} u_0\| \, \|u_0\| < \epsilon \quad \Rightarrow$ regularity."

Note:

$$\|u_\lambda(t)\|^2 = \frac{1}{\lambda}\|u(\lambda^2 t)\|^2 \ - \ \text{"subcritical"}$$

$$\|A^{\frac{1}{2}} u_\lambda(t)\|^2 = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{1}{2}} \lambda u(\lambda^2 t, \lambda x)|^2 \mathrm{d}x = \lambda \|A^{\frac{1}{2}} u(\lambda^2 t)\|^2 - \text{"supercritical"}$$

$$\|A^{\frac{1}{2}} (u_0)_\lambda\| \, \|(u_0)_\lambda\| = \|A^{\frac{1}{2}} u_0\| \, \|u_0\| \ - \ \text{"critical"}$$

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Self-similar solutions:

Definition

u is a <u>self-similar solution</u> if $u(x,t) = u_\lambda(x,t)\ \forall\ \lambda > 0$.

In Fourier space:

$$u(x,t) = \lambda u(\lambda x, \lambda^2 t)\ \Rightarrow\ \widehat{u}(\xi,t) = \frac{1}{\lambda^2}\widehat{u}\left(\frac{\xi}{\lambda}, \lambda^2 t\right).$$

Can self-similar Soln's help understand the role of scaling in NSE problem?

Effect of self-similar scaling.

NSE in mild formulation in Fourier space:

$$\widehat{u}(\xi,t) = \widehat{u}_0(\xi)\mathrm{e}^{-|\xi|^2 t} + \int_0^t \mathrm{e}^{-|\xi|^2 s} \frac{|\xi|}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \widehat{u}(\eta, t-s) \odot_{e_\xi} \widehat{u}(\xi-\eta, t-s)\, \mathrm{d}\eta\, \mathrm{d}s,$$

with $\xi \cdot \widehat{u}(\xi) = 0$, $\widehat{u}(-\xi) = \overline{\widehat{u}(\xi)}$, $e_\xi = \xi/|\xi|$,

$$v \odot_{e_\xi} w = -i\,(e_\xi \cdot w)\,\pi_{e_\xi^\perp} v.$$

In self-similar case, re-scale using $\widehat{u}(\xi,t) = \frac{1}{\lambda^2}\widehat{u}\left(\frac{\xi}{\lambda}, \lambda^2 t\right)$ with $\lambda = |\xi|$:

$$\widehat{u}(e_\xi, |\xi|^2 t) = \widehat{u}_0(e_\xi)\, \mathrm{e}^{-|\xi|^2 t}$$
$$+ \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\tau \int_{\mathbb{R}^3} \mathrm{e}^{-|\xi|^2 s} \widehat{u}(e_\eta, |\eta|^2(t-s)) \odot_\xi \widehat{u}(e_{\xi-\eta}, |\xi-\eta|^2(t-s)) \frac{|\xi|^3 \mathrm{d}\eta \mathrm{d}s}{|\eta|^2 |\xi-\eta|^2}.$$

After a change of vars. (with $\tau = |\xi|^2 t$ – "similarity horizon")...

Self-Similar rescaling of NSE

For $\tau = |\xi|^2 t$:

$$\widehat{u}(e_\xi, \tau) = \widehat{u}_0(e_\xi)\, e^{-\tau}$$

$$+\frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^\tau \underbrace{e^{-\sigma}}_{Exp(1)} \int_{\mathbb{R}^3} \widehat{u}(e_\eta, |\eta|^2(\tau-\sigma)) \odot_\xi \widehat{u}(e_{e_\xi-\eta}, |e_\xi-\eta|^2(\tau-\sigma)) \underbrace{\frac{d\eta d\sigma}{|\eta|^2|e_\xi-\eta|^2}}_{\pi^3 H(\eta|e_\xi) d\eta d\sigma}$$

In non-ss case the same re-scaling $v(\xi, t) = c\dfrac{\widehat{u}(\xi, t)}{|\xi|^2}$ leads to:

$$v(\xi, t) = v_0(\xi) e^{-|\xi|^2 t} + \int_0^t |\xi|^2 e^{-|\xi|^2 s} \int_{\mathbb{R}^3} v(\eta, t-s) \odot_{e_\xi} v(\xi-\eta, t-s)\, H(\eta|\xi)\, d\eta\, ds,$$

$$H(\eta|\xi) = \frac{|\xi|}{\pi^3 |\eta|^2 |\xi-\eta|^2}.$$

Probabilistic interpretation.

Abuse notation $\widehat{u} \rightsquigarrow v$...

$$v(e_\xi, \tau) = v_0(e_\xi)\,\mathrm{e}^{-\tau} +$$

$$+ \int_0^\tau \mathrm{e}^{-\sigma} \int_{\mathbb{R}^3} v(e_\eta, |\eta|^2(\tau-\sigma)) \odot_{e_\xi} v(e_{e_\xi-\eta}, |e_\xi-\eta|^2(\tau-\sigma)) \frac{\mathrm{d}\eta \mathrm{d}\sigma}{\pi^3|\eta|^2|e_\xi-\eta|^2}$$

"Solution" process:

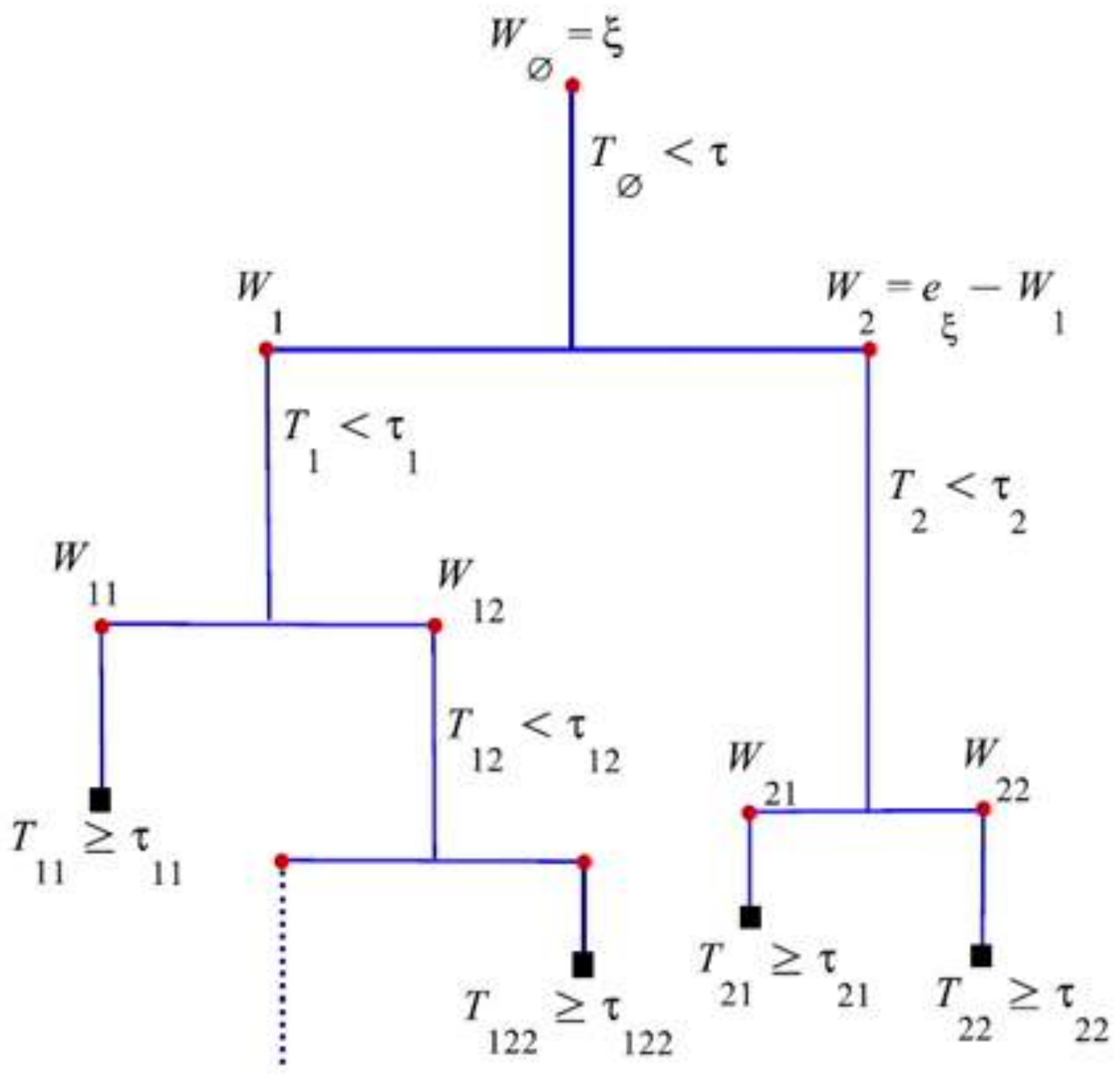
$$X(e,\tau) = v_0(e)\,\mathbb{1}_{T_\emptyset \geq \tau} + X^{(1)}(e_{W_1}, \tau_1) \odot_e X^{(2)}(e_{W_2}, \tau_2)\,\mathbb{1}_{T_\emptyset < \tau},$$

where $T_\emptyset \sim Exp(1)$, $W_1 \sim H(\cdot|e)$, $W_2 = e - W_1$, $\tau_j = |W_j|^2(\tau - T_\emptyset)$.

If $\mathbb{E}(|X(e_\xi,\tau)|) < \infty$, then:

$v(e_\xi, \tau) = \mathbb{E}(X(e_\xi, \tau))$ is a solution to ssmNSE.

Self-Similar Cascade.



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Note: if $\tau_{121} \geq T_{121}$ (the dotted branch in the picture ends):

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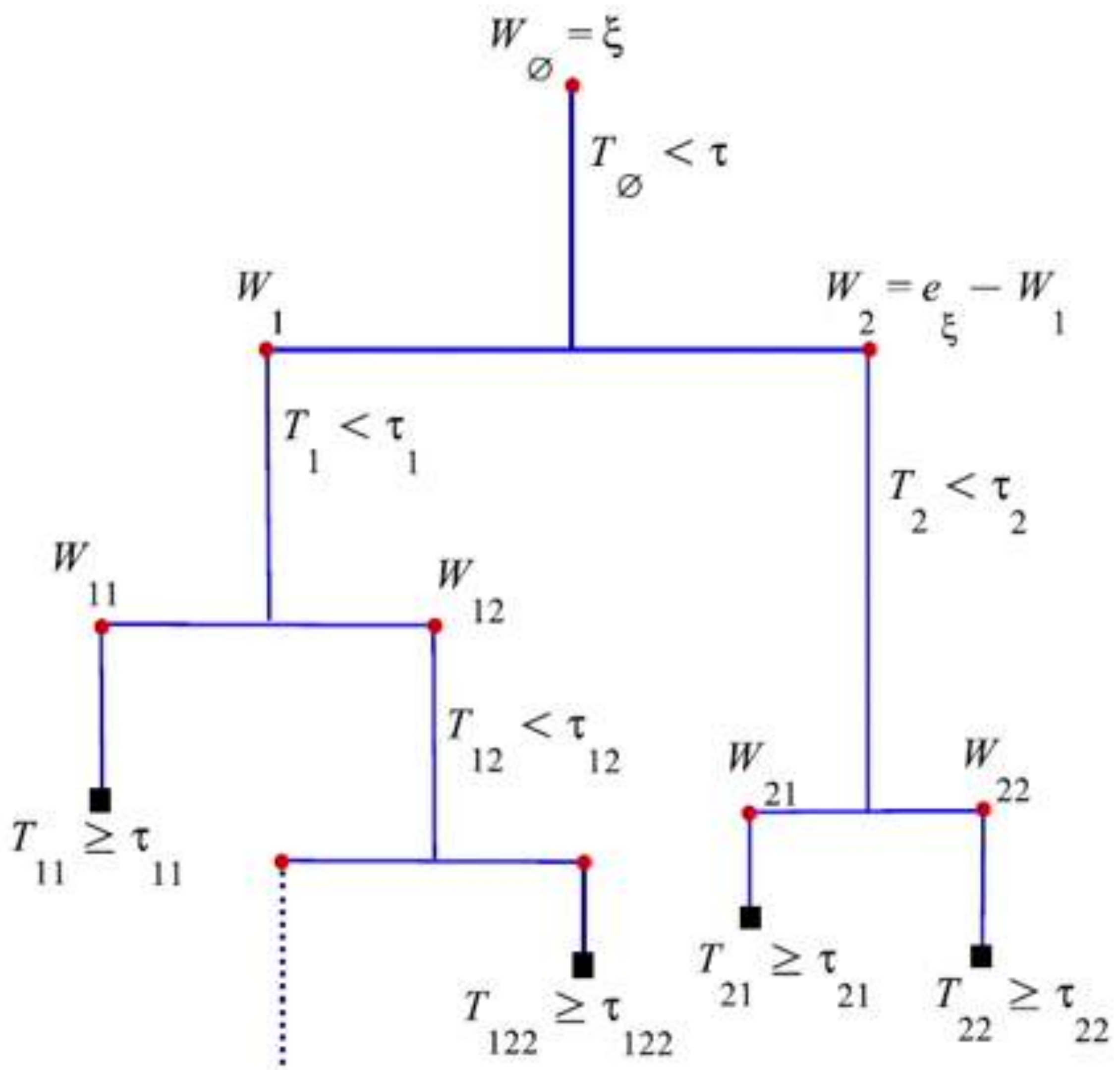

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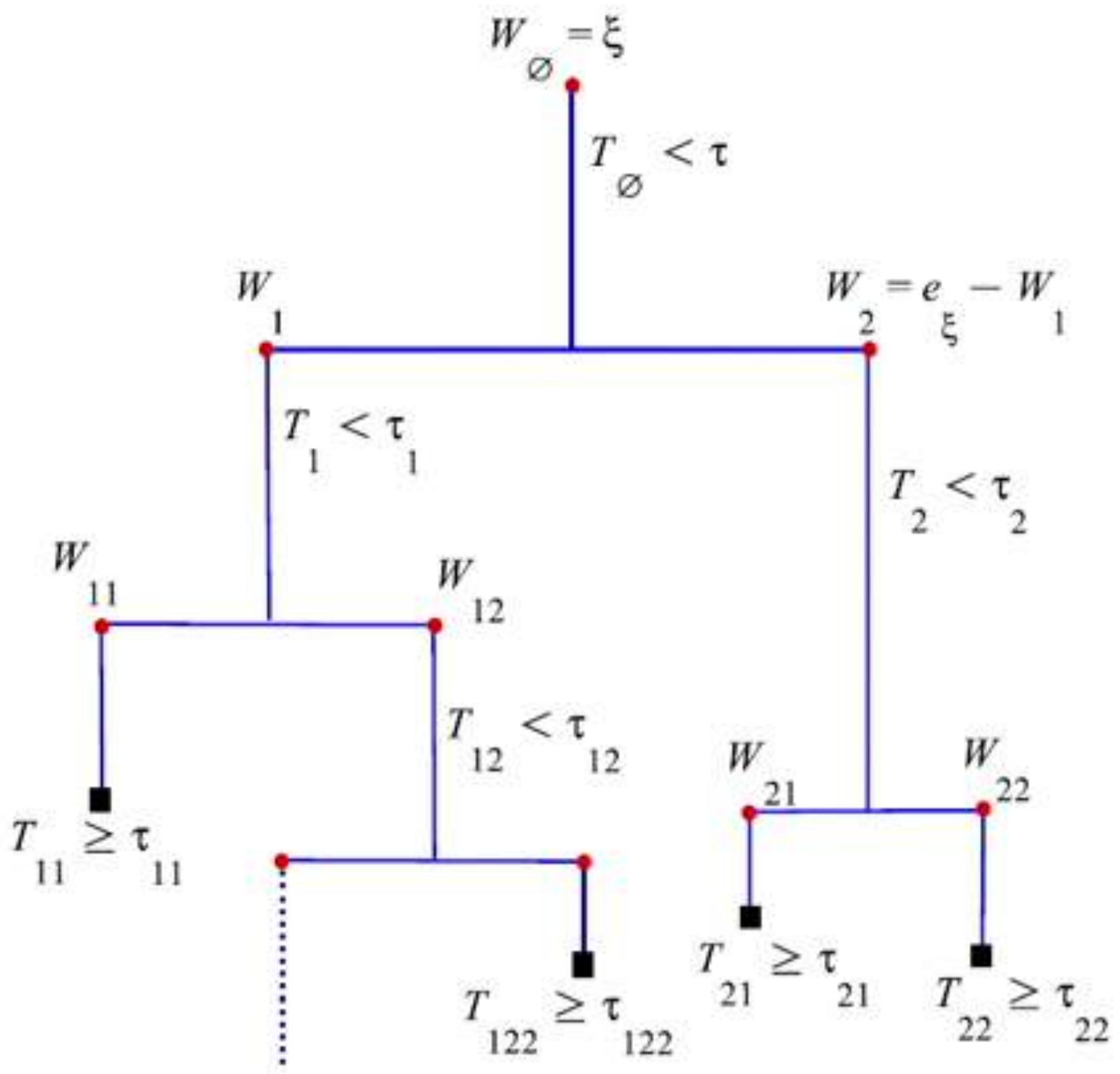

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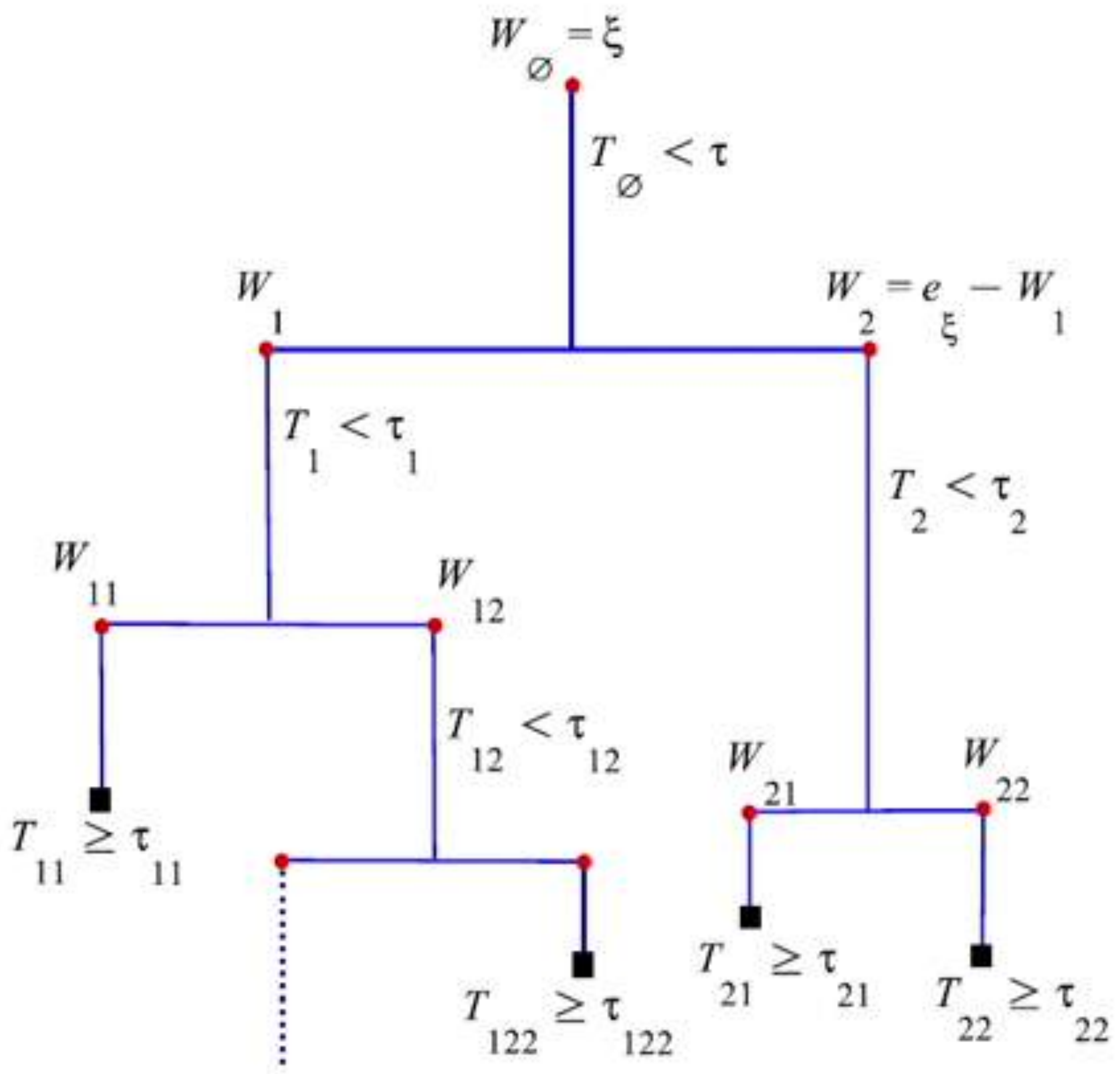

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Recall: cascade continues at $b \in \{1,2\}^n$ if $\tau_b - T_b > 0$. But:

$$\tau_b - T_b = |W_b|^2(\tau_{b|n-1} - T_{b|n-1}) - T_b = \dots$$
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$\Rightarrow$ "explosion horizon / shortest path" R.V:

$$S(e_\xi) = \lim_{n\to\infty} \inf_{|b|=n} \underbrace{\sum_{j=0}^{n} \frac{T_{b|j}}{\prod_{k=0}^{j} |W_{b|k}|^2}}_{\theta_b}$$

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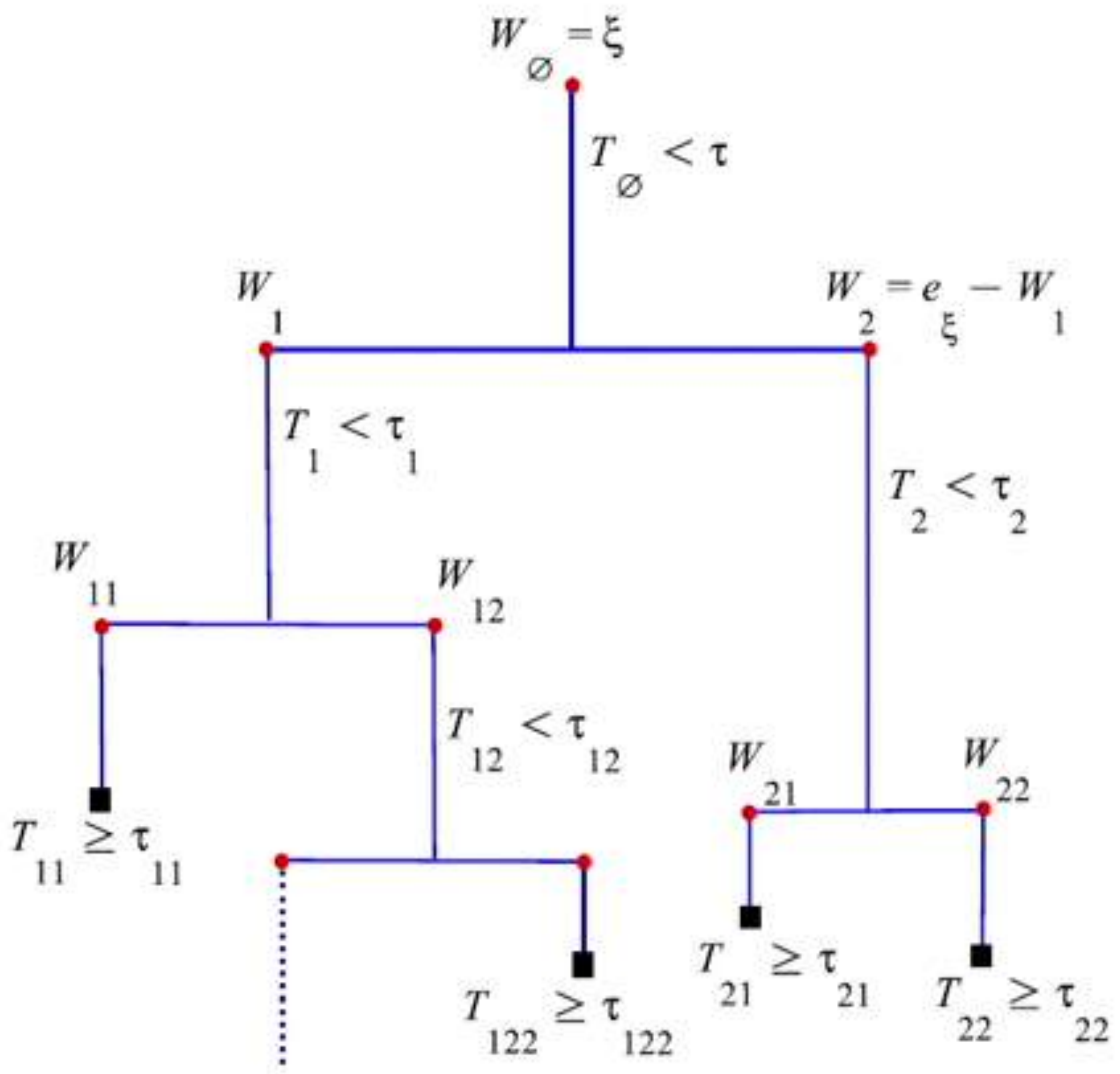

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Simplifications:

- $H(\eta_1|e) \rightsquigarrow \delta_{\sqrt{\alpha}}$
- Ignore geometry: $\odot \rightsquigarrow$ product of magnitudes.
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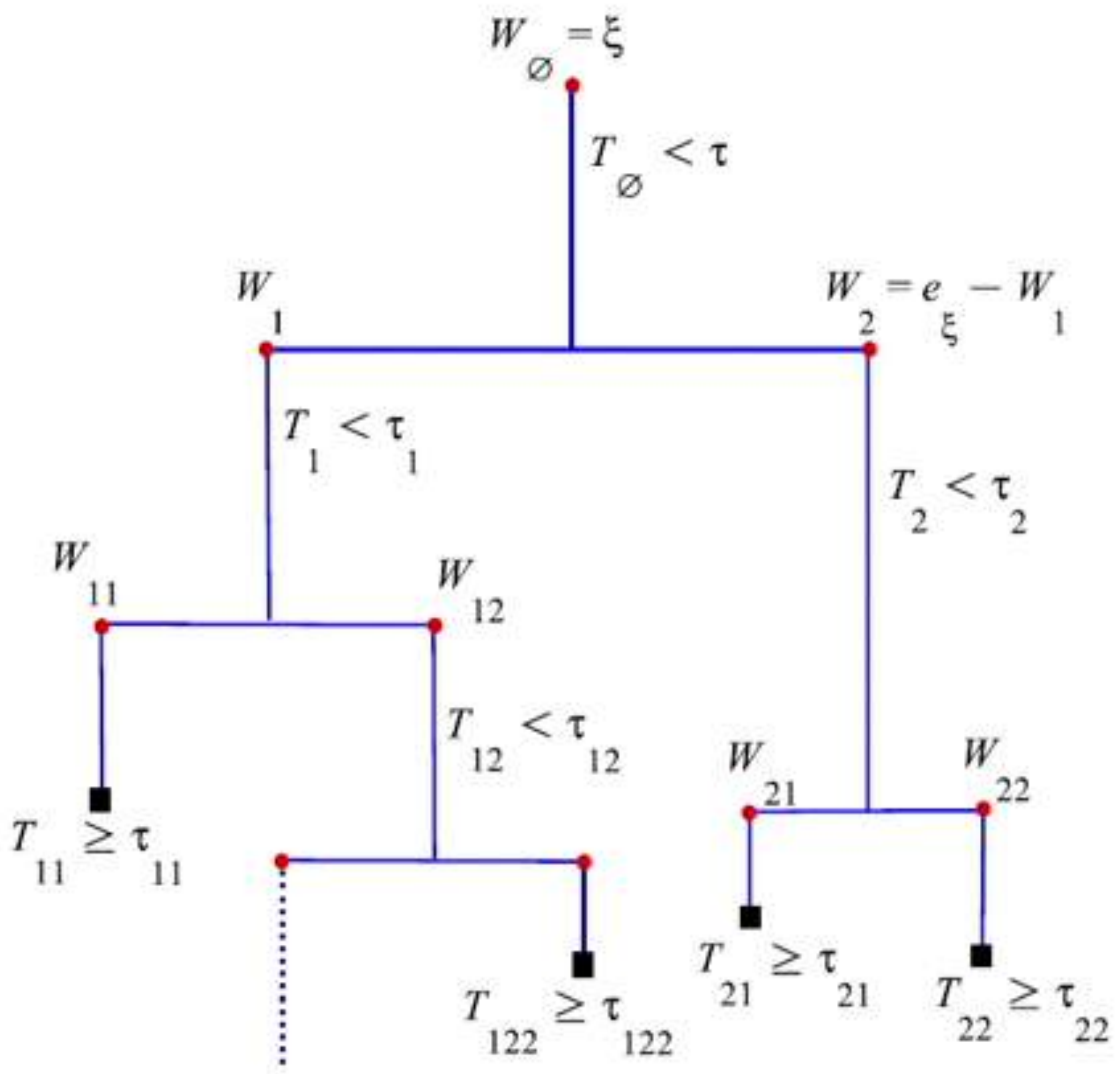

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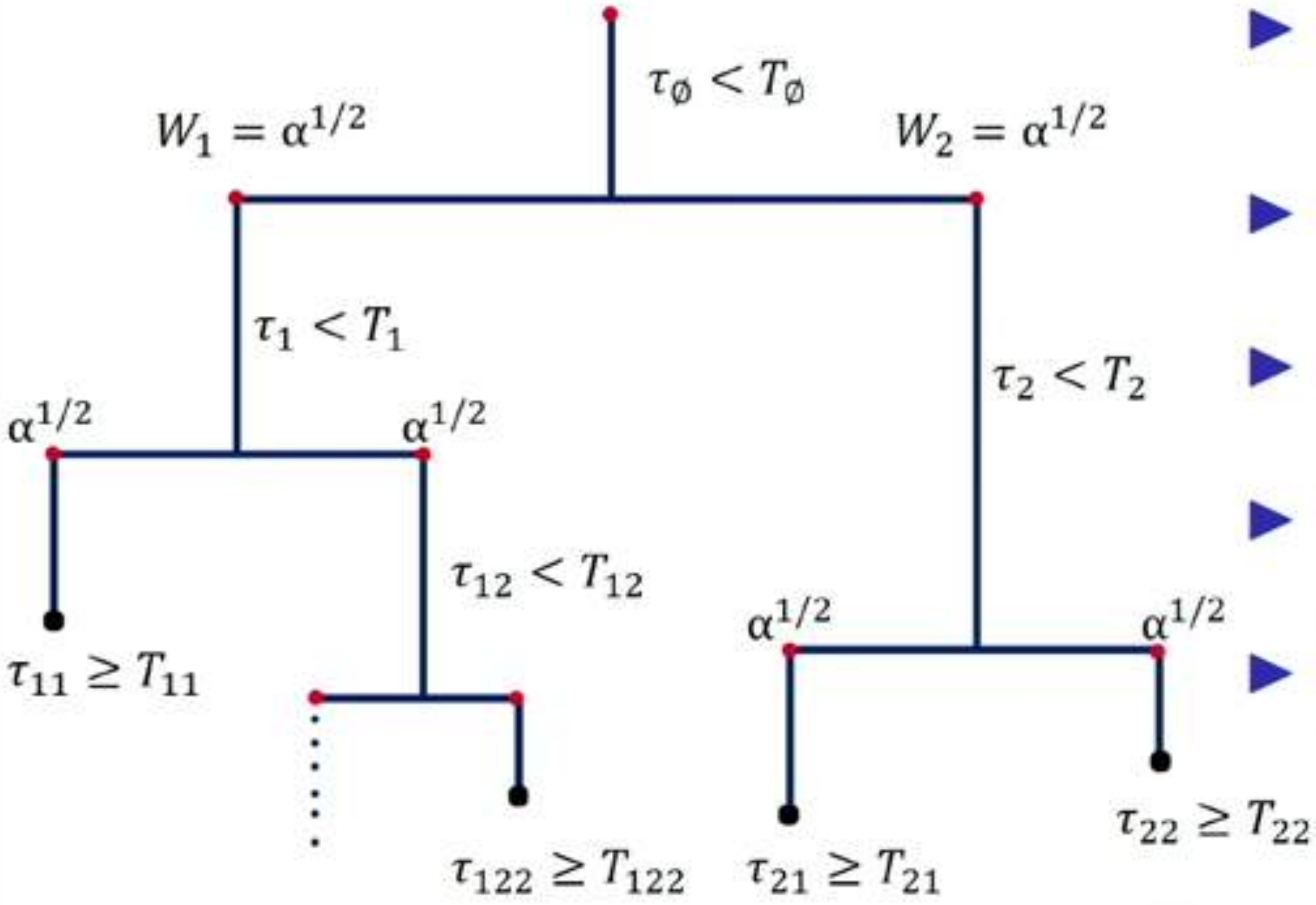
Cascade set-up for α-Riccati:

$$v(\tau) = v_0\, e^{-\tau} + \int_0^\tau e^{-\sigma} v^2(\alpha(\tau - \sigma))\, d\sigma.$$

Solution Process = Process with *inherited property*:

$$X(\tau) = X(0)\, \mathbb{1}_{T_\emptyset \geq \tau} + X^{(1)}(\alpha\tau_1) X^{(2)}(\alpha\tau_2)\, \mathbb{1}_{T_\emptyset < \tau},$$



- $\mathbb{E}(|X(\tau)|) < \infty \Rightarrow v(\tau) = \mathbb{E}(X(\tau))$ a solution, $v_0 = \mathbb{E}(X(0))$.
- T_b – i.i.d. Exp(1).
- $\tau_{bj} = \alpha(\tau_b - T_b)$, $\tau_\emptyset = \tau$.
- $T_b \geq \tau_b$ – process at branch b stops.
- Non-explosion $\Rightarrow$ uniqueness.

Note: if $\tau_{121} \geq T_{121}$: $X(\tau_\emptyset) = v_0^5$.

Some inherited events

Define:

Shortest path: $S = \lim_{n\to\infty} \min_{|b|=n} \sum_{k=0}^{n} \frac{T_{b|k}}{\alpha^k}$,

Longest path: $L = \lim_{n\to\infty} \max_{|b|=n} \sum_{k=0}^{n} \frac{T_{b|k}}{\alpha^k}$,

explosion events: $\{S < \infty\}$ – explosion; $\{L < \infty\}$ – *hyper-explosion*.

Note:

- no-explosion and hyper-explosion are inherited events.
- $X(t) = \mathbb{1}_{S\geq\tau}$, $X(t) = \mathbb{1}_{L<\tau}$ – processes with inherited property.

Conclusion:
$v = \mathbb{P}(S = \infty)$, $v = \mathbb{P}(L < \infty)$, $v(\tau) = \mathbb{P}(S \geq \tau)$, $v(\tau) = \mathbb{P}(L < \tau)$
solve α-Riccati.

The case $\alpha \leq 1$.

Theorem

When $\alpha \in [0,1]$ *the cascade in non-exploding:* $\mathbb{P}(S = \infty) = 1$.

Proof.

Compare to Yule process. □

Consequence: $X(t) = v_0^{N(\tau)}$, $N(\tau)$–number of branches;

Theorem

Let $\alpha < 1$. $\forall\ v_0 > 0\ \exists!$ *solution* $v(t) = \mathbb{E}(v_0^{N(\tau)}) < \infty$*:*

$$\lim_{n\to\infty} \mathbb{E}(v_0^{N(\tau)}) = \begin{cases} 0, & 0 \leq v_0 < 1, \\ 1, & v_0 = 1, \\ \infty, & v_0 > 1 \end{cases} .$$

Proof.

Estimate $\mathbb{P}(N(\tau) = n)$. □

Special Cases.

- $\alpha = 0$: equation is $v'(\tau) = -v(\tau) + v_0^2$.

$\mathbb{P}(N(\tau) = 1) = \mathrm{e}^{-\tau}, \quad \mathbb{P}(N(\tau) = 2) = 1 - \mathrm{e}^{-\tau},$

$$v(\tau) = \mathbb{E}(v_0^{N(\tau)}) = v_0 \mathrm{e}^{-\tau} + v_0^2(1 - \mathrm{e}^{-\tau}).$$

- $\alpha = 1$: equation is $v'(\tau) = -v(\tau) + v^2(\tau)$.

$\mathbb{P}(N(\tau) = n) = \mathrm{e}^{-\tau}(1 - \mathrm{e}^{-\tau})^{n-1}, \quad n \in \mathbb{N}.$

$$v(\tau) = \mathbb{E}(v_0^{N(\tau)}) = \sum_{n=1}^{\infty} v_0^n \mathrm{e}^{-\tau}(1 - \mathrm{e}^{-\tau})^{n-1} = \frac{v_0 \mathrm{e}^{-\tau}}{1 - v_0(1 - \mathrm{e}^{-\tau})}.$$

- $\alpha = 1/2$: equation is $v'(\tau) = -v(\tau) + v^2(\tau/2)$.

$\mathbb{P}(N(\tau) = n) = \mathrm{e}^{-\tau}\tau^{n-1}/(n-1)!, \quad n \in \mathbb{N}.$

$$v(\tau) = \mathbb{E}(v_0^{N(\tau)}) = \sum_{n=1}^{\infty} v_0^n \frac{\tau^{n-1}}{(n-1)!} \mathrm{e}^{-\tau} = v_0 \mathrm{e}^{v_0 \tau - 1}.$$

The case $\alpha > 1$: explosion.

Theorem (Maximal-path Explosion Theorem)

Suppose $\exists\ C < 1$ s.t. in the self-similar cascade for any b, $Z_b = \max\{|W_{b1}|, |W_{b2}|\}$ satisfies $\mathbb{E}(Z_b^{-2}) \leq C$. Then the cascade is exploding: $\mathbb{P}(S < \infty) = 1$.

...In the α-Riccati case $|W_b| = \alpha\ \forall\ b$, so the cascade is exploding when $\alpha > 1$.

In fact:

Theorem

When $\alpha > 1$ the cascade for α-Riccati is hyper-exploding: $\mathbb{P}(L < \infty) = 1$.

Question: How to build process $X(t)$ in the presence exploding branches?

$\alpha > 1$. Iterative Process.

Start with an $X_0(\tau) \geq 0$, and set

$$X_n(\tau) = \begin{cases} v_0, & T_\emptyset \geq \tau, \\ X^{(1)}_{n-1}(\alpha(\tau - T_\emptyset))\, X^{(2)}_{n-1}(\alpha(\tau - T_\emptyset)), & T_\emptyset < \tau \end{cases}, \quad n \in \mathbb{N},$$

If $y_n(\tau) = \mathbb{E}(X_n(\tau)) < \infty$:

$$y_n(\tau) = u_0 e^{-\tau} + \int_0^\tau e^{-\sigma} y^2_{n-1}(\alpha(\tau - \sigma))\, \mathrm{d}\sigma\,,$$

Question: For which X_0 we can pass to the limit?

$\alpha > 1$. Minimal Solution: Existence.

Choose $X_0 = 0$. Then $X_n(\tau) \to \underline{X}(\tau) = v_0^{N(t)} \mathbb{1}_{S>t}$.

Note: $v_0 = 1$, $\underline{X}(\tau) = \mathbb{1}_{S>t} \to 0, \tau \to \infty$; $v_0 = 0$, $\underline{X} = 0$ a.s.

Theorem

- *If $\underline{v}(\tau) = \mathbb{E}(\underline{X}(\tau)) < \infty$, then $\underline{v}(\tau)$ is a solution;*
- *if $v(\tau)$ is another solution for the same initial data, then $\underline{v}(\tau) < v(\tau)$.*

Theorem

Let $\alpha > 1$.

- *If $v_0 \leq \max\{1, (2\alpha - 1)/4\}$, then $\underline{v}(\tau) = \mathbb{E}(\underline{X}(\tau))$ is a (global) solution.*
- *If $v_0 > 2\alpha - 1$ then local solutions blow up in finite time.*

$\alpha > 1$: Uniqueness Problem for $0 \le v_0 \le 1$.

Theorem

Let $0 \le v_0 < 1$. Then $\underline{v}(\tau)$ is the unique solution in the class $\|v(\tau)\|_\infty < 1$.

Proof.

Let $v(\tau)$ be a solution. Initialize iterations with $X_0 = v(\tau)$ and compare to $\underline{X}$... □

"Upper Solution": use $X_0 = 1$: $X_n(\tau) \to \overline{X}(\tau) = v_0^{N(\tau)}$. Set $\overline{v}(\tau) = \mathbb{E}(\overline{X}(\tau))$.

Note: $v_0 = 0$, $\overline{X}(\tau) = \mathbb{1}_{L \le t} \to 1$, $\tau \to \infty$; $v_0 = 1$, $\overline{X} = 1$ a.s.

Theorem

Let $\alpha > 1$ and $v_0 \in [0, 1]$. Three are at least two solutions:

- $\underline{v}(\tau) \to 0$ *as* $\tau \to \infty$,
- $\overline{v}(\tau) \to 1$ *as* $\tau \to \infty$.

Moreover, if $v(\tau) \in [0, 1]$, then $\underline{v}(\tau) \le v(\tau) \le \overline{v}(\tau)$.

$\alpha > 1$. Iterative Process.

Start with an $X_0(\tau) \geq 0$, and set

$$X_n(\tau) = \begin{cases} v_0, & T_\emptyset \geq \tau, \\ X_{n-1}^{(1)}(\alpha(\tau - T_\emptyset))\, X_{n-1}^{(2)}(\alpha(\tau - T_\emptyset)), & T_\emptyset < \tau \end{cases}, \quad n \in \mathbb{N},$$

If $y_n(\tau) = \mathbb{E}(X_n(\tau)) < \infty$:

$$y_n(\tau) = u_0 e^{-\tau} + \int_0^\tau e^{-\sigma} y_{n-1}^2(\alpha(\tau - \sigma))\, d\sigma ,$$

Question: For which X_0 we can pass to the limit?

$\alpha > 1$: Uniqueness Problem for $0 \leq v_0 \leq 1$.

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Let $0 \leq v_0 < 1$. Then $\underline{v}(\tau)$ is the unique solution in the class $\|v(\tau)\|_\infty < 1$.

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Note: $v_0 = 0$, $\overline{X}(\tau) = \mathbb{1}_{L \leq t} \to 1$, $\tau \to \infty$; $\quad v_0 = 1$, $\overline{X} = 1$ a.s.

Theorem

Let $\alpha > 1$ and $v_0 \in [0, 1]$. Three are at least two solutions:

- $\underline{v}(\tau) \to 0$ *as* $\tau \to \infty$,
- $\overline{v}(\tau) \to 1$ *as* $\tau \to \infty$.

Moreover, if $v(\tau) \in [0, 1]$, then $\underline{v}(\tau) \leq v(\tau) \leq \overline{v}(\tau)$.

$\alpha > 1$. Non-uniqueness for big initial data.

Theorem

Let $\alpha > 5/2$ *and* $1 < u_0 \leq \frac{2\alpha-1}{4} - \phi_\infty$, ($\phi_\infty(\alpha) < 3/4$). *Then*

$$\underline{v}(\tau) < \overline{v}(\tau) \leq \underline{w} + (1 - e^{-\tau}),$$

where $\underline{w}$ *is the minimal solution corresponding to initial data* $w_0 \in [v_0 + \phi_\infty, (2\alpha - 1)/4]$.

Also can be shown: local in time solutions are not unique.

A Random Initialization.

$$X_0(t) = \begin{cases} 0, & T_\emptyset \geq \tau, \\ G(\tau - T_\emptyset), & T_\emptyset < \tau, \end{cases}$$

Denote

$$F(t) := \mathbb{E}(X_0(\tau)) = \int_0^\tau e^{-\sigma} G(\tau - \sigma)\, d\sigma \quad \text{or,} \quad \left\| \begin{array}{l} F'(\tau) + F(\tau) = G(\tau) \\ F(0) = 0. \end{array} \right.$$

If $L_n := \max_{|b|=n} \underbrace{\sum_{k=0}^{n} \frac{T_{b|k}}{\alpha^k}}_{\theta_b} < \tau$ then $\quad X_n(\tau) = \prod_{|\tau|=n} G(\alpha^n(\tau - \theta_b)).$

and so

$$\mathbb{E}\left(X_{n+1}(\tau)\, \mathbb{1}_{L_{n+1}<\tau} \mid T_b,\ |b| \leq n\right) = \prod_{|b|=n} F^2(\alpha^{n+1}(\tau - \theta_b)).$$

Note: if $F^2(\alpha\tau) \leq G(\tau)$, then $X_n(\tau)\, \mathbb{1}_{L_n<\tau}$ is a super-martingale.

Athreya's Solution.

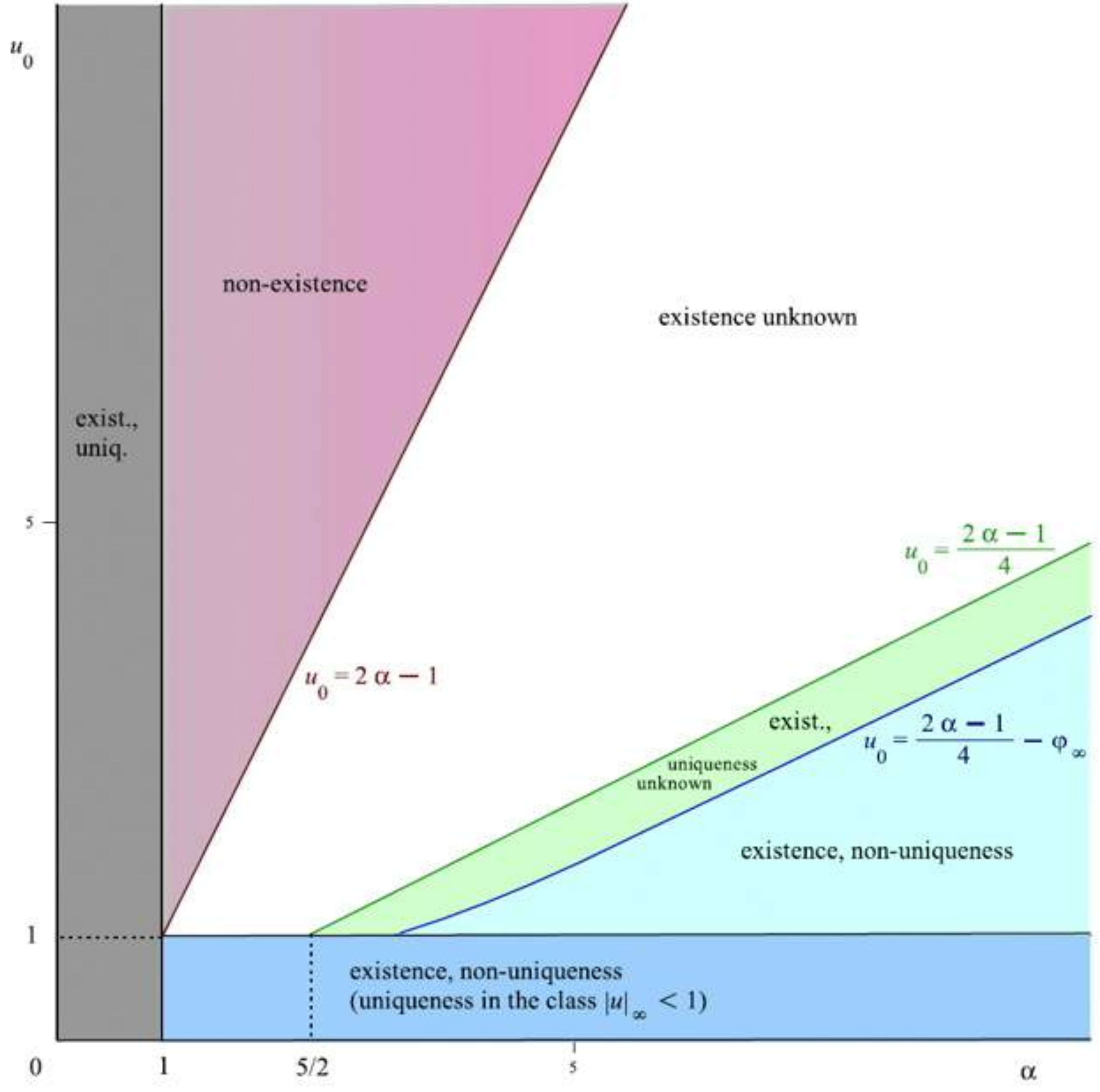
For $\alpha > 1$ and $v_0 = 0$, Choose

$$F(\tau) = e^{-\tau^{-b}}, \qquad b = \frac{\ln 2}{\ln \alpha}.$$

Then $X_n(\tau) = X_n(\tau)\mathbb{1}_{L_n<\tau}$ is an *uniformly integrable* super-martingale.

Conclusion: $\exists\, X(t) = \lim_n X_n(t)$ and $v_A(t) = \mathbb{E}(X(t))$ is yet another solution to α-Riccati.

Existence&Uniqueness regions for α-Riccati.



Back to NSE: explosion?

Some properties of $H(\eta|e_\xi) = \frac{1}{\pi^3}\frac{1}{|\eta|^2|e_\xi - \eta|}$:

- Rotational invariance;
- $\forall\, b$, $\{|W_{b|j}|\}_j$ – iid $\frac{2}{\pi^2}\frac{1}{r}\ln\left|\frac{1+r}{1-r}\right|$.
- In polar coordinates at each branching b:
  - azimuth angle for W_{b1} is Unif$(0, 2\pi)$;
  - longitudinal angles for W_{b1} and W_{b2} are jointly Unif$\{0 \le \phi_1, \phi_2 \le \pi,\ \phi_1 + \phi_2 \le \pi\}$.

Consequence:

$$\max\{|W_{b1}|, |W_{b2}|\} \quad \text{is distributed} \quad z(r) = \frac{4}{\pi}\frac{1}{r}\ln\left|\frac{r}{r-1}\right|.$$

Note:

$$\text{if } Z \sim z(r), \quad \text{then} \quad \mathbb{E}(Z^{-2}) = \frac{2}{\pi} < 1.$$

Thus, by the Maximal-Path Explosion Theorem ssmNSE cascade is exploding!

Maximal-Path Explosion

Recall the Maximal-Path Explosion Theorem:

Theorem

Suppose $\exists$ $C < 1$ *s.t. in the self-similar cascade for any b,* $Z_b = \max\{|W_{b1}|, |W_{b2}|\}$ *satisfies* $\mathbb{E}(Z_b^{-2}) \leq C$*. Then the cascade is exploding* $\mathbb{P}(S < \infty) = 1$.

Idea of Proof:

Recall: Explosion event $\{S = \infty\}$, where $S = \lim_{n\to\infty} \min_{|b|=n} \sum_{j=0}^{n} \frac{T_{b|j}}{\prod_{k=0}^{j} |W_{b|k}|^2}$

Consider "maximal-descendant" path $b_m \in \{1, 2\}^{\mathbb{N}}$ that at each junction $b = b_m|j$ follows the path of maximal $|W_{bi}|$, $i = 1, 2$. Note:

$$S \leq \sum_{j=0}^{\infty} \frac{T_{b_m|j}}{\prod_{k=0}^{j} |W_{b_m|k}|^2} \quad \text{a.s.}$$

What could be proven for NSE?

Existence and uniqueness for v_0 via minimal solution:

Theorem
Suppose $\|v_0(e_\xi)\|_\infty < 1$. Then there exists a unique ssmNSE solution in the class $\|v(e_\xi, \tau)\|_\infty < 1$.

(In fact existence can be proven for $\|v_0(e_\xi)\|_\infty \leq C$ for a $C > 1$.)

Conclusions/Challenges

- Can iterative procedure be used to prove non-uniqueness for NSE? Lack of hyper-explosion is a challenge.
- Can we prove existence for NSE for arbitrary large initial data?

  Good: NSE nonlinearity – $\odot$-product along a tree – contains repeated projection of the same vector on random planes, which should deplete non-linearity.

  Bad: A large tree generates long $\odot$-products: enough v_0 may survive the depletion and make X large.
- Can this explosion approach be connected to Kolmogorov backwards equation... Develop a general theory of explosion-nonuniqueness for nonlinear PDE?
- Can cascade structure be employed to describe energy transfer between scales with consequences in turbulence/regularity?